Solution to the Riemann-Hilbert boundary value problem for multiply connected domains

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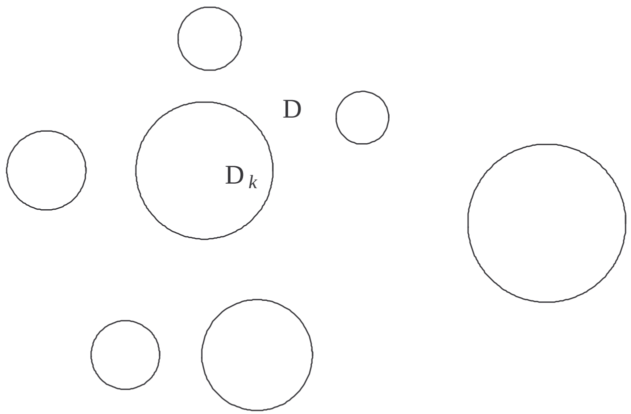
1. Riemann-Hilbert problem for multiply connected domains

Consider mutually disjoint disks $D_k = \{z \in \mathbb{C}: |z - a_k| < r_k\}$, $k = 1, 2, \ldots, n$, in the complex plane $\mathbb{C}$, $D = \mathbb{C} \setminus \bigcup_{k=1}^{n} (D_k \cup \partial D_k)$.



Given Hölder continuous functions $\lambda(t) \neq 0$ and $h(t)$. To find a function $\phi(z)$ analytic in D continuous in $D \cup \partial D$ with the following boundary condition

$$\mathrm{Re}\,\overline{\lambda(t)}\,\phi(t) = h(t) \text{ on } |t - a_k| = r_k,\ k = 1, 2, \ldots, n. \qquad (\mathrm{RH})$$

The problem (1.1) is a partial case of the $\mathbb{R}$-linear problem problem

$$\phi^+(t) = a(t)\,\phi^-(t) + b(t)\,\overline{\phi^-(t)} + c(t) \text{ on } |t - a_k| = r_k,\quad k = 1, 2, \ldots, n. \qquad (\mathbb{R})$$

History of the problem (RH): (sorry, not all is cited)

RH Riemann (1851) Statement and discussion of the vector-matrix problem $\Phi^+ = G\,\Phi^-$ and a note about determination of analytic function by a relation between its **Re** and **Im** on the boundary.

θ Poincaré (1882) Automorphic function and θ-series.

RH Volterra (1883) Statement and first simple results.

θ H. F. Baker (1897) "Abel's theorem and the allied theory", Cambridge (new edition 1995)

RH Hilbert (1904, 1924) Reduction to singular integral equation with the kernel ctg $\frac{\sigma-s}{2}$.

RH Villat & Dini (1912) Solution in closed form to the Dirichlet problem for annulus.

$\mathbb{R}$ Muskhelishvili (1932), I. N. Vekua & Ruhadze (1933) (antiplane elastic problem, transmision problem, perfect contact)

Sch Golusin (1934) A functional equation for multiply connected domains and generalized alternating Schwarz method.

RH Gakhov (1941) Closed form solution for simply connected domains.

RH Kveselava (1945) Reduction to integral equations. Preminaliry result on Bojarski's system.

ℝ Markushevich (1946) Statement of the $\boldsymbol{R}$ –linear problem in complex form.

RH I. N. Vekua (1952) Integral equations, estimation of the deffect numbers.

"Generalized analytic functions", Nauka, Moscow, 1988.

RH Bojarski (1958-1959) Estimation of the deffect numbers. Solvability of the problem had been reduced to solvability of the linear algebraic system with analytic coefficients (Bojarski's system). Special case $0 \le \chi \le n$.

ℝ Bojarski (1960) "elliptic case"of the $\boldsymbol{R}$ –linear problem (see L. G. Mikhajlov (1963)) .

RH Muskhelishvili "Singular integral equations", Nauka, Moscow, 1968.

Sch Mikhlin (1949) Generalized alternating Schwarz method (Decomposition methods) and Schwarz's operator.

RH Zverovich (1971) Using of Riemann surfaces.

θ T. Akaza, K. Inoue (1984) Limit sets and Poincaré series for Schottky groups.

RH Sch θ $\mathbb{R}$ + Functional Eqs $\Longrightarrow$ Complete solution to the problem: Mityushev (1993-1998)

"Constructive methods to linear and non-linear boundary value problems for analytic function. Theory and applications", Chapman & Hall / CRC, 1999 (with S. V. Rogosin)

Auxiliary

C_A - Banach space of functions analytic in $\bigcup_{k=1}^{n} D_k$ and continuous in $\bigcup_{k=1}^{n} (D_k \bigcup \partial D_k)$ with the norm $\| f \| = \max_{k=1,2,\dots,n} \max_{|t-a_k|=r_k} | f(t) |$.

$z^*{}_{(k)} := \frac{r_k{}^2}{\overline{z - a_k}} + a_k$ - inversion with respect to the circle $|\, t - a_k \,| = r_k$.

If $f(z)$ is analytic in $|\, z - a_k \,| < r_k$, then $\overline{f(z^*{}_{(k)})}$ is analytic in $|\, z - a_k \,| > r_k$.

Introduce $z^*{}_{(k_m\, k_{m-1} \ldots\, k_1)} := (z^*{}_{(k_{m-1}\, k_{m-2} \ldots\, k_1)})^*_{k_m}$, which can be written in the form

$\gamma_j(z) = \frac{e_j\, z + b_j}{c_j\, z + d_j}$, $m \in 2\,\mathbb{Z}$, $\gamma_j(\bar{z}) = \frac{e_j\, \bar{z} + b_j}{c_j\, \bar{z} + d_j}$, $m \in 2\,\mathbb{Z} + 1$, $e_j d_j$-$b_j c_j = 1$, m - level of γ_j

$\gamma_0(z) = z$, $\gamma_1(\bar{z}) = z^*{}_{(1)}$, $\gamma_2(\bar{z}) = z^*{}_{(2)}$, ..., $\gamma_n(\bar{z}) = z^*{}_{(n)}$, $\gamma_{n+1}(z) = z^*{}_{(21)}$, $\gamma_{n+2}(z) = z^*{}_{(31)}$,...

$\mathcal{K} := \{\gamma_j(z)$ of even level$\}$, $\mathcal{F} := \{\gamma_j(\bar{z})$ of odd level$\}$

2. Harmonic measures

Let $w \in D$ be a fixed point. To find a function $\alpha_s(z) = \alpha_s(x, y)$ $(z = x + i\, y)$ harmonic in D continuous in $D \bigcup \partial D$ with the following boundary condition

$$\alpha_s(t) = \delta_{s\,l} \text{ on } |\, t - a_k \,| = r_k, \; k = 1, 2, \; ..., n\,, \tag{2.1}$$

where $\delta_{s\,l}$ is the Kronecker symbol. Represent the harmonic function $\alpha_s(z)$ in the form

$$\alpha_s(z) = \mathrm{Re}\,\phi(z) + \sum_{m=1}^{n} A_m \ln |\, z - a_m \,| + A, \tag{2.2}$$

where $\sum_{m=1}^{n} A_m = 0, \; \phi(w) = 0.$

1. (2.1) $\Longleftrightarrow$ R-linear problem: $\phi(t) = \phi_k(t) - \overline{\phi_k(t)} + f(t), \; |\, t - a_k \,| = r_k, \; k = 1, 2, \; ..., n\,,$

where $$f(z) = \delta_{s\,l} - \sum_{m \neq k} A_m \ln |\, z - a_m \,| + A_k \ln r_k + A \text{ in } |\, z - a_k \,| \le r_k. \tag{2.3}$$

Introduce

$$\Phi(z) = \begin{cases} \phi_k(z) + \sum_{m\neq k} [\overline{\phi_m(z^*_{(m)})} - \overline{\phi_m(w^*_{(m)})}] - \overline{\phi_k(w^*_{(k)})} + f(z), & |z - a_k| \leq r_k, \\ & k = 1, 2, \ldots, n, \\ \phi(z) + \sum_{m=1}^{n} [\overline{\phi_m(z^*_{(m)})} - \overline{\phi_m(w^*_{(m)})}], & z \in D. \end{cases}$$

$$|t - a_k| = r_k: \quad \Phi^+(t) - \Phi^-(t) = \phi(t) - \phi_k(t) + \overline{\phi_k(t)} - f_k(t) = 0 \Longrightarrow \Phi(z) = \phi(w) = 0$$

2. $\Longrightarrow$ system of functional equations

$$\phi_k(z) = -\sum_{m\neq k} [\overline{\phi_m(z^*_{(m)})} - \overline{\phi_m(w^*_{(m)})}] + \overline{\phi_k(w^*_{(k)})} - f(z), \quad |z - a_k| \leq r_k, \; k = 1, 2, \ldots, n, \tag{2.4}$$

$$\phi(z) = -\sum_{m=1}^{n} [\overline{\phi_m(z^*_{(m)})} - \overline{\phi_m(w^*_{(m)})}], \quad z \in D. \tag{2.5}$$

Lemma. Given $f(z) \in C_A$ and numbers $\mu_k \in [0, 2\pi)$. The system of functional equations

$$\phi_k(z) = -e^{i\mu_k} \sum_{m \neq k} e^{-i\mu_m} [\overline{\phi_m(z^*_{(m)})} - \overline{\phi_m(w^*_{(m)})}] + \overline{\phi_k(w^*_{(k)})} - f(z), \quad (2.6)$$

$$|z - a_k| \leq r_k, k = 1, 2, \ldots, n,$$

has a unique solution in C_A. This solution can be found by the method of successive approximations. The approximations converges in C_A.

The proof of Lemma is based on solvability of the ℝ-linear problem Bojarski (1960)

Application of Lemma to (2.4) and (2.5) yields

$$\phi(z) = -\sum_{m=1}^{n} A_m \Big(\sum_{k \neq m} \ln \frac{\overline{z^*_{(k)} - a_m}}{\overline{w^*_{(k)} - a_m}} + \sum_{k \neq m,\, k_1 \neq k} \ln \frac{z^*_{(k\,k_1)} - a_m}{w^*_{(k\,k_1)} - a_m} + \ldots \Big) =$$

$$\sum_{m=1}^{n} A_m \ln \prod_{j=1,\, j \neq m}^{\infty} \psi_m^j(z), \quad \text{where } \psi_m^j(z) = \begin{cases} \dfrac{\gamma_j(z) - a_m}{\gamma_j(w) - a_m} & \text{if} \quad \gamma_j \in \mathcal{K}_m, \\ \overline{\left(\dfrac{\gamma_j(\overline{z}) - a_m}{\gamma_j(\overline{w}) - a_m}\right)} & \text{if} \quad \text{if } \gamma_j \in \mathcal{F}_m, \end{cases}$$

$\mathcal{K}_m$ (or $\mathcal{F}_m$) $= \{z^*{}_{(k_j\, k_{j-1} \ldots k_1)} \in \mathcal{K} : \; k_j \neq m\}$.

Harmonic measures

$$\alpha_s(z) = \sum_{m=1}^{n} A_m \ln | \prod_{j \in \mathcal{K}_m \bigcup \mathcal{F}_m}^{\infty} \psi_m^j(z) | + A_0 .$$

The real constants A_m $(m = 0,\ 1,\ \ldots,\ n)$ satisfy a linear algebraic system. They are written explicitly.

3. Schwarz's operator

Complex Green's function is represented in the form

$$M(z,\zeta) = M_0(z,\zeta) + \sum_{m=1}^{n} \alpha_m(\zeta)\ln(z-a_m) - \ln(\zeta - z) + A(\zeta), \tag{3.1}$$

where $M_0(z,\zeta)$ is a single-valued analytic function of z in D (for any fixed ζ), $M_0(w,\zeta) = 0$, $A(\zeta)$ is unknown. The boundary value problem for $M_0(z,\zeta)$

$$\mathrm{Re}[M_0(t,\zeta) + \sum_{m=1}^{n} \alpha_m(z)\ln(t-a_m) - \ln(\zeta - t) + A(\zeta)] = 0, \; |t-a_k| = r_k, \;\; k = 1, 2, \; \ldots, n \tag{3.2}$$

is reduced to the system of functional equations

$$\phi_k(z) = -\sum_{m\neq k} [\overline{\phi_m(z^*_{(m)})} - \overline{\phi_m(w^*_{(m)})}] - f(z), \quad |z-a_k| \leq r_k, \;\; k = 1, 2, \; \ldots, n. \tag{3.3}$$

Application of Lemma to (3.3) yields the formula

$$M_0(z,\zeta) = \sum_{m=1}^{n} \alpha_m(z) \ln \prod_{j=1,\, j\neq m}^{\infty} \psi_m^j(z) + \omega(z,\zeta), \qquad (3.4)$$

where $\omega(z,\zeta) = \ln \prod_{j=1}^{\infty} \omega_j(z,\zeta)$, $\omega_j(z,\zeta) = \begin{cases} \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)} & \text{if} \quad \gamma_j \in \mathcal{K}, \\ \frac{\overline{\zeta-\gamma_j(\overline{z})}}{\overline{\zeta-\gamma_j(\overline{w})}} & \text{if} \quad \text{if } \gamma_j \in \mathcal{F}. \end{cases}$

Schwarz's operator solves the problem $\operatorname{Re}\phi(t) = f(t)$ on $|t - a_k| = r_k$, $k = 1, 2, \ldots, n$ with single-valued $\operatorname{Re}\phi(z)$ in D ($\operatorname{Im}\phi(z)$ is multi-valued in D). It is written explicitly in the following form

$$\phi(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{|\zeta-a_k|=r_k} f(\zeta)\times$$

$$\left(\sum_{\gamma_j\in\mathcal{K}} \left(\frac{1}{\zeta-\gamma_j(w)} - \frac{1}{\zeta-\gamma_j(z)}\right) + \left(\frac{r_k}{\zeta-a_k}\right)^2 \sum_{\gamma_j\in\mathcal{F}} \left(\frac{1}{\overline{\zeta-\gamma_j(\overline{z})}} - \frac{1}{\overline{\zeta-\gamma_j(\overline{w})}}\right) - \frac{1}{\zeta-z}\right) d\zeta +$$

$$\frac{1}{2\pi i} \sum_{k=1}^{n} \int_{|\zeta-a_k|=r_k} f(\zeta)\, \frac{\partial A}{\partial \nu}(\zeta)\, d\sigma + \sum_{k=1}^{n} A_k[\ln(z-a_k) + \psi_k(z)] + i\varrho.$$

It is possible to write explicitly $A(\zeta)$.

4. Riemann-Hilbert problem

Using the factorization of $\lambda(t)$ reduce the problem (1.1) to

$$\mathrm{Re}\, e^{-i\mu_k}\, \omega(t) = g(t) \text{ on } |t - a_k| = r_k,\ k = 1, 2, \ldots, n. \qquad (4.1')$$

<u>Remark</u>. It is possible to write explicitly $e^{-i\mu_k}$, $g(t)$ and the factorization function $X(z)$. The Riemann-Hilbert problem (4.1') is reduced to the functional equations

$$\phi_k(z) = -e^{-i\mu_k} \sum_{m\neq k} e^{i\mu_m} \left[\overline{\phi_m(z^*_{(m)})} - \overline{\phi_m(w^*_{(m)})}\right] + \overline{\phi_k(w^*_{(k)})} + f(z),$$

$$|z - a_k| \le r_k,\ k = 1, 2, \ldots, n, \qquad (4.2)$$

where $f(z)$ has the following structure $f(z) = h(z) + \sum_{s=1}^{2\chi} p_s\, \beta_s(z) + e^{-i\mu_k} Q$, $h(z)$ and $\beta_s(z)$ are known, p_s are arbitrary real constants (for positive index χ of $\lambda(t)$), Q is arbitrary complex constant. Note that $\beta_s(z)$ does not depend on $g(t)$ which enters into $h(z)$.

The solution of (4.2) has the form

$\phi_k(z) = (\mathcal{A}_k\, h)\,(z) + \sum_{s=1}^{2\,\chi} p_s(\mathcal{A}_k\, \beta_s)\,(z) + e^{-i\mu_k}\, Q + c$, where the operator $\mathcal{A}_k$ is defined by the formula

$$(\mathcal{A}_k\, F)\,(z) = e^{-i\mu_k} \sum_{m=0}^{\infty} (-1)^m \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \cdots$$
$$\sum_{k_m \neq k_{m-1}} \mathbb{C}^m\, e^{-i\mu_{k_m}} [F(z^*{}_{(k_m\, k_{m-1} \ldots\, k_1)}) - F(w^*{}_{(k_m\, k_{m-1} \ldots\, k_1)})] + F(z)$$

$| z - a_k | \;\leq r_k,$

$$\omega(z) = Q - \sum_{m=1}^{n} e^{i\mu_k} [\overline{\phi_m(z^*{}_{(m)})} - \overline{\phi_m(w^*{}_{(m)})}] \tag{4.3}$$

Theorem 1 ($\chi \geq 0$) **Let at least two numbers μ_k are non-equal. Then the Riemann-Hilbert problem (4.1) is solvable iff the linear algebraic system with respect to p_s and Q is solvable**

$$\mathrm{Re}[-e^{-i\mu_k} \sum_{m \neq k} e^{i\mu_m}[(\mathcal{A}_k f)(w^*_{(m\,k)}) - (\mathcal{A}_k f)(w^*_{(m)})] - h(w^*_{(k)}) - \sum_{s=1}^{\chi} p_s \beta_s(w^*_{(k)}) + e^{-i\mu_k} Q] = 0,$$

$$k = 1, 2, \ldots, n. \tag{4.4}$$

If (2.6) is fulfilled, the general solution has the form

$$\phi(z) = \prod_{m=1}^{n} (z - a_k)^{\chi_k} [X(z)\,\omega(z) + \sum_{s=1}^{\chi} \delta_s z^s]. \tag{4.5}$$

In (4.5) χ undetermined complex constants δ_s are transformed into 2 χ real p_s.

Existence of the system (4.4) were predicted by Bojarski (see addition to I.N. Vekua Generalized analytic functions, Nauka, Moscow, 1988).

Theorem 2 ($\chi < 0$) Let at least two numbers μ_k are non-equal. Then the Riemann-Hilbert problem (4.1) is solvable iff for some Q

$$\mathrm{Re}[-e^{-i\mu_k} \sum_{m \neq k} e^{i\mu_m}[(\mathcal{A}_k f)(w^*_{(m\,k)}) - (\mathcal{A}_k f)(w^*_{(m)})] - h(w^*_{(k)}) + e^{-i\mu_k} Q] = 0,$$

$$k = 1, 2, \ldots, n.$$

If (2.6) is fulfilled, the general solution has the form

$$\phi(z) = \prod_{m=1}^{n} (z - a_k)^{\chi_k} X(z)\, \omega(z).$$

The solution $\phi(z)$ is regular at infinity iff $\omega(z)$ has zero at $z = \infty$ of order $|\chi|$.

4. Poincaré series θ

Let $H(z)$ be a meromorphic function. The Poincaré θ_{2q} −series

$$\theta_{2q}(z) := \sum_{\gamma_j \in \mathcal{K}} H[\gamma_j(z)]\,(c_j z + d_j)^{-2q} \tag{4.6}$$

is associated with the group $\mathcal{K}$.

When $q > 1$ the series (4.6) converges absolutely and almost uniformly in a subset of $\mathbb{C}$. When $q = 1$ the series (4.6) can be either absolutely convergent or absolutely divergent. Poincaré proposed to investigate the absolute convergence by comparison with the series $\sum_{j=0}^{\infty} |c_j|^{-2}$. Necessary and sufficient conditions for absolute convergence of the series have been found by [Akaza T., Inoue K. Limit sets of geometrically finite free Kleinian groups, Tohoku Math. J., 36, 1-16, 1984].

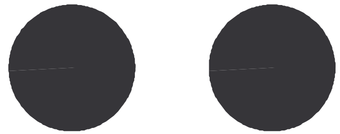
However, the series $\theta_2(z)$ is always uniformly convergent. More precisely [Mityushev V. Convergence of the Poincare series for classical Schottky groups, Proc. AMS, 126, 8, 2399-2406, 1998]

Theorem 3 . Let a rational function $H(z)$ has poles only at regular points of $\mathcal{K}$. Then the Poincaré θ_2- series converges uniformly in every compact subset od D / {limit points of $\mathcal{K}$}.

Proof is based on the representation $\theta_2(z) = -\frac{1}{2}[(\phi(z) + H(z)) - (\omega(z) - H(z))]$, where $\phi(z)$ and $\omega(z)$ have the form of (2.5), where $\phi_k(z)$ satisfy similar functional equations.

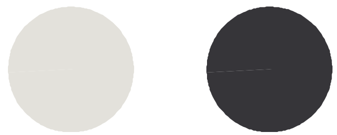
5. Generalizid method of Schwarz Sch

Dirichlet problem: $u(t) = f(t)$ on $|t - a_k| = r_k,\ k = 1, 2.$



First approximation:

Second approximation:



Integral equations corresponding to the method

$$u_k(z) = -\frac{1}{2\pi} \sum_{m\neq k} \int_{L_m} u_m(\tau)\left[\frac{\partial G_m(z,\tau)}{\partial n} - \frac{\partial G_m(w,\tau)}{\partial n}\right] ds + \frac{1}{2\pi} \int_{L_k} u_k(\tau) \frac{\partial G_k(w,\tau)}{\partial n} ds + f(z),$$
$$k = 1, 2, \ \ldots, n.$$

$G_m(z, \tau)$ is Green's function of the exterior of D_m.