Darcy flow around a two–dimensional fracture

Vladimir Mityushev[1] and Pierre M. Adler[2]

[1]Dept.Math., Pedagogical University,
ul. Podchorazych 2, Krakow 30-084, Poland,
[2] IPGP, tour 24, 4, place Jussieu,
75252 - Paris Cedex 05, France

E-mail: adler@ipgp.jussieu.fr

Abstract. The flow in and around a fracture modeled as a two–dimensional permeable lens immersed in an infinite porous medium of different permeability is analytically solved by means of conformal mapping and Fourier transform. The flow singularities near the angular points of the lens have been determined. Predictions are successfully compared with data obtained by numerical code.

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1. Introduction

Fractures are of great practical importance since they can drastically influence flows through porous media with a small permeability [1]. A fracture can be considered as a void space between two solid surfaces, but very often real fractures are filled with debris; therefore, they can be themselves considered as porous media where Darcy law applies.

In the recent years, attention has been mostly focused on the numerical solution of flow through fracture networks [6] and fractured porous media [2], [3]. Very little attention has been given to analytical solutions. For instance, flow in and around a single ellipsoid can be found in [4].

The main purpose of this paper is to solve the Darcy equations in and around a single fracture modelled as a two–dimensional lens filled by a porous medium of permeability K_i' embedded in an infinite porous medium of permeability K_e'. These equations will be solved by the complex potential method which was systematically applied to two–dimensional problems [7]. Use of conformal mapping reduces the boundary value problems to problems for canonical domains. The latter problems are usually solved in closed form.

This paper is organized as follows. Section 2 is devoted to a general presentation of the physical situation and of the Darcy equations to be solved.

The solution by means of conformal mapping and Fourier transform is detailed in Section 3 when the flow at infinity is parallel to the lens. The solution to a number equation and the representation of the meromorphic function as an infinite sum of simple fractions (see Appendix) is essentially used here.

Of special interest is the flow near the angular points of the lens. Similar problems for corners were discussed by Obnosov [9] and Keller [5] who have constructed only radial solutions. The singularities of the present flow field are extracted analytically in Section 5.

The solution when the flow at infinity is perpendicular to the lens is briefly addressed in Section 4 where the analytical solutions are compared to numerical determination of the flow and pressure field.

Some concluding remarks end this paper.

2. General

In the complex plane $\mathbb{C}$, consider the domain D_i bounded by two arcs Γ_1 and Γ_2 of the two circles $|z' + \mathbf{i}b'| = r_0'$ and $|z' - \mathbf{i}b'| = r_0'$, where $z' = x' + \mathbf{i}y'$, $\mathbf{i} = \sqrt{-1}$ (see Figure 1).

Let D_e be the complement of D to $\mathbb{C} \cup \infty$. It is convenient to introduce the points a' and $-a'$ lying on the real axis where the arcs Γ_1 and Γ_2 meet with the angle $\pi\alpha$. If a' and $\pi\alpha$ are known, $b' = a' \cot \frac{\pi\alpha}{2}$, $r_0' = a' / \sin \frac{\pi\alpha}{2}$. Let the domains D_i and D_e be occupied by media of permeabilities K_i' and K_e', respectively. The flow velocity v_β' satisfies the

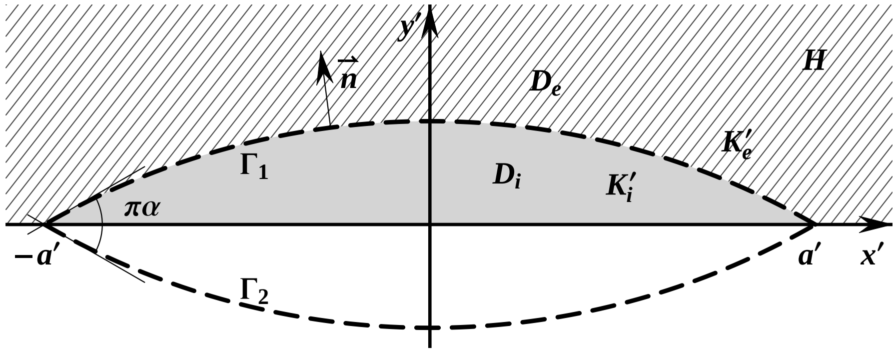

Figure 1. The two–dimensional permeable lens.

Darcy equation where the subscript β stands for i or e, μ' is the fluid viscosity [1]

$$\mathbf{v}'_\beta = -\frac{K'_\beta}{\mu'}\nabla p'_\beta, \quad \nabla' \cdot \mathbf{v}'_\beta = 0. \tag{1}$$

The pressure and normal fluxes are continuous on the surface $\Gamma_1 \cup \Gamma_2$

$$p' = p'_e, \tag{2}$$

$$K'_i \mathbf{n} \cdot \nabla' p'_i = K'_e \mathbf{n} \cdot \nabla' p'_e, \tag{3}$$

where $\mathbf{n}$ is the unit normal to $\Gamma_1 \cup \Gamma_2$.

The flow is generated by a constant pressure gradient $\overline{\nabla' p'}$ applied at infinity; therefore,

$$p'_e(\mathbf{x}') \sim \mathbf{x}' \cdot \overline{\nabla' p'}, \quad \text{as} \quad |\mathbf{x}'| \to \infty \tag{4}$$

or equivalently,

$$\mathbf{v}'_e = -\frac{K'_e}{\mu'}\overline{\nabla' p'}, \quad \text{as} \quad |\mathbf{x}'| \to \infty. \tag{5}$$

It is more convenient to work with dimensionless quantities which are denoted by the same letters as the dimensional quantities, but without any prime. Let us define

$$K = \frac{K'_i}{K'_e}, \quad \mathbf{x} = \frac{\mathbf{x}'}{a'}, \quad p'_\beta(\mathbf{x}) = \frac{p'_\beta}{|\overline{\nabla' p'}| a'}, \quad \mathbf{v}_\beta = -\frac{K'_\beta |\overline{\nabla' p'}|}{\mu'} \nabla p_\beta.$$

The dimensionless fields p_β and $\mathbf{v}_\beta$ verify the dimensionless equations

$$\mathbf{v}_e = -\nabla p_e, \quad \mathbf{v}_i = -K\nabla p_i, \quad \nabla \cdot \mathbf{v}_\beta = 0 \quad (\beta = i, e) \tag{6}$$

with the boundary conditions

$$p_i = p_e, \tag{7}$$

$$K\mathbf{n} \cdot \nabla p_i = \mathbf{n} \cdot \nabla p_e, \quad \text{on} \quad \Gamma_1 \cup \Gamma_2, \tag{8}$$

$$p_e \sim \mathbf{x} \cdot \overline{\nabla p}, \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{9}$$

The dimensionless gradient $\overline{\nabla p}$ has two components $p_{,1\infty}$ and $p_{,2\infty}$. Equation (9) can be written in term of the velocity

$$\mathbf{v}_e(x,y) = (v_1(x,y), v_2(x,y)) \sim -(p_{,1\infty}, p_{,2\infty}), \quad \text{as} \quad x, y \to \infty. \tag{10}$$

The functions $p_i(x,y)$ and $p_e(x,y)$ are harmonic in the domains D_i and D_e, respectively, continuously differentiable in the closures of the considered domains, except the points $z = \pm 1$, where $p(x,y)$ and $p_e(x,y)$ are bounded (here $z = x + \mathbf{i}y$). The velocities $\mathbf{v}_i$ and $\mathbf{v}_e$ are continuous in the closure of the considered domains, except $z = \pm 1$, where they may have an integrable singularity.

Following [8], the two complex potentials $\varphi_i(z)$ and $\varphi_e(z)$ can be introduced

$$p_i(x,y) = \frac{2}{K+1} Re\varphi_i(z) \quad \text{and} \quad p_e(x,y) = Re\varphi_e(z), \tag{11}$$

where Re stands for the real part. The function $\varphi_i(z)$ is analytic in D_i; $\varphi_e(z)$ is analytic in D_e, except infinity, where (see (9))

$$\varphi_e(z) \sim (p_{,1\infty} - ip_{,2\infty})z, \quad \text{as} \quad z \to \infty. \tag{12}$$

The functions $\varphi_i(z)$ and $\varphi_e(z)$ satisfy the $\mathbb{R}$–linear problem [8]

$$\varphi_e(t) = \varphi_i(t) - \rho\overline{\varphi_i(t)}, \quad t \in \Gamma_1 \cup \Gamma_2, \tag{13}$$

where ρ has the form

$$\rho = \frac{K-1}{K+1}. \tag{14}$$

In order to simplify calculations, it is convenient to decompose the problem. In the first problem, the pressure gradient is parallel to the x–axis. Therefore, $\varphi_i(z)$ and $\varphi_e(z)$ are symmetric with respect to the x–axis

$$\varphi_i(z) = \overline{\varphi_i(\overline{z})}, \quad \varphi_e(z) = \overline{\varphi_e(\overline{z})} \tag{15}$$

and

$$\varphi_e(z) \sim z, \quad \text{as} \quad z \to \infty. \tag{16}$$

In the second problem, $\overline{\nabla p}$ is parallel to the y–axis. Therefore, the complex potentials of the anti-symmetric with respect to the x–axis problem, $\varphi_i^*(z)$ and $\varphi_e^*(z)$, satisfy the relations

$$\varphi_i^*(z) = -\overline{\varphi_i^*(\overline{z})}, \quad \varphi_e^*(z) = -\overline{\varphi_e^*(\overline{z})} \tag{17}$$

and

$$\varphi_e^*(z) \sim -iz, \quad \text{as} \quad z \to \infty. \tag{18}$$

3. Flow parallel to the lens

The $\mathbb{R}$–linear problem (13) is solved in two steps. First, we use a conformal mapping and obtain a problem in a strip. Second, we apply the Fourier transform to solve the latter problem in closed form. Here, we consider the symmetric case (eq:1.7)–(16).

3.1. Conformal mapping

The conformal mapping

$$w = \ln \frac{z+1}{z-1} + \pi \mathbf{i} \tag{19}$$

transforms the upper half–plane H onto the horizontal strip $0 < \eta < \pi$, where $w = \xi + \mathbf{i}\eta$; moreover, $D_i \cap H$ is transformed onto $0 < \eta < \frac{\pi\alpha}{2}$, and $D_e \cap H$ onto $\frac{\pi\alpha}{2} < \eta < \pi$. The inverse conformal mapping has the form

$$z = \frac{e^w - 1}{e^w + 1}. \tag{20}$$

The corresponding domains are illustrated in Figure 2.

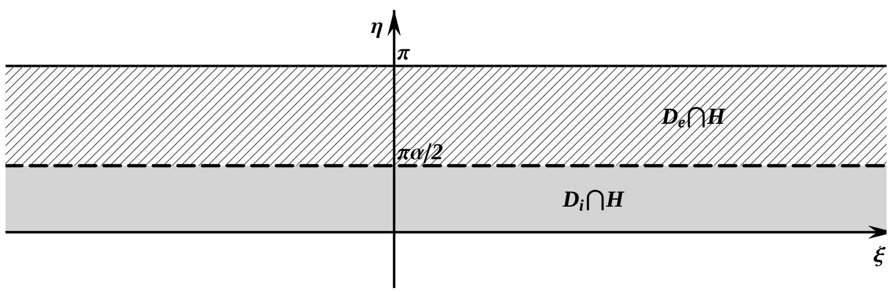

Figure 2. The image of the physical plane under the conformal mapping.

Introduce the functions

$$\Phi(w) = \varphi_i \left(\frac{e^w - 1}{e^w + 1} \right), \quad \Phi_e(w) = \varphi_e \left(\frac{e^w - 1}{e^w + 1} \right). \tag{21}$$

They are analytic in the strips $0 < \eta < \frac{\pi\alpha}{2}$ and $\frac{\pi\alpha}{2} < \eta < \pi$, respectively. The point $z = \infty$ transforms to the point $w = \pi\mathbf{i}$ under (19). This implies that (16) is replaced by

$$\Phi_e(w) \sim \frac{e^w - 1}{e^w + 1} \sim \frac{2}{w - \pi\mathbf{i}}, \quad \text{as} \quad w \to \pi\mathbf{i}, \tag{22}$$

i.e., $\Phi_e(w)$ has a simple pole at $w = \pi\mathbf{i}$ with the residue 2. Hence, we can represent $\Phi_e(w)$ in the form

$$\Phi_e(w) = \Omega(w) + \frac{2}{w - \pi\mathbf{i}}. \tag{23}$$

The functions $\Phi(w)$ and $\Omega(w)$ are analytic in $0 < \eta < \frac{\pi\alpha}{2}$ and $\frac{\pi\alpha}{2} < \eta < \pi$, respectively; They are continuously differentiable in the closures of the considered domains except at the points $w = \pm\infty$, where they are bounded. Taking into account (21) and (23), (13) implies that

$$\Omega(w) = \Phi(w) - \rho\overline{\Phi(w)} - \frac{2}{w - \pi\mathbf{i}}, \quad w = \xi + \mathbf{i}\frac{\pi\alpha}{2}. \tag{24}$$

The symmetry conditions (15) on the real axis yield the conditions

$$Im\Omega(w) = 0, \quad w = \xi + \pi\mathbf{i}, \tag{25}$$

$$Im\Phi(w) = 0, \quad w = \xi, \tag{26}$$

where Im stands for the imaginary part. Consider now the boundary value problem (24)–(26) with respect to functions $\Phi(w)$ and $\Omega(w)$ analytic in $0 < \eta < \frac{\pi\alpha}{2}$ and $\frac{\pi\alpha}{2} < \eta < \pi$, respectively and continuous in the closures of the considered strips. Moreover, $\Phi(w)$ and $\Omega(w)$ are bounded at infinity, i.e., when $\xi \to \pm\infty$, $0 \le \eta \le \pi$ where $w = \xi + \mathbf{i}\eta$.

3.2. Fourier transform

The boundary value problem (24)–(26) is solved by applying the Fourier transform. Represent $\Phi(w)$ and $\Omega(w)$ via real harmonic functions

$$\Phi(w) = u_1(\xi,\eta) + \mathbf{i}v_1(\xi,\eta), \quad \Omega(w) = u_2(\xi,\eta) + \mathbf{i}v_2(\xi,\eta). \tag{27}$$

Following [8], the problem (24)–(26) is rewritten as

$$u_2\left(\xi, \frac{\pi\alpha}{2}\right) = (1-\rho)u_1\left(\xi, \frac{\pi\alpha}{2}\right) - \frac{2\xi}{\xi^2 + \pi^2(1-\frac{\alpha}{2})^2}, \tag{28}$$

$$\frac{\partial u_2}{\partial \eta}\left(\xi, \frac{\pi\alpha}{2}\right) = (1+\rho)\frac{\partial u_1}{\partial \eta}\left(\xi, \frac{\pi\alpha}{2}\right) - \frac{4\pi(1-\frac{\alpha}{2})\xi}{\left(\xi^2 + \pi^2(1-\frac{\alpha}{2})^2\right)^2}, \tag{29}$$

$$\frac{\partial u_1}{\partial \eta}(\xi, 0) = 0, \tag{30}$$

$$\frac{\partial u_2}{\partial \eta}(\xi, \pi) = 0. \tag{31}$$

Here, we differentiate the imaginary part of (24) with respect to ξ

$$\frac{\partial v_2}{\partial \xi}\left(\xi, \frac{\pi\alpha}{2}\right) = (1+\rho)\frac{\partial v_1}{\partial \xi}\left(\xi, \frac{\pi\alpha}{2}\right) - \frac{\partial}{\partial \xi}\left(\frac{2\pi(1-\frac{\alpha}{2})}{\xi^2 + \pi^2(1-\frac{\alpha}{2})^2}\right) \tag{32}$$

and apply the Cauchy–Riemann equation $\frac{\partial v_2}{\partial \xi} = -\frac{\partial u_2}{\partial \eta}$. Then (32) yields (29).

Introduce the Fourier transform on the variable ξ

$$U_j(\eta) = U_j(\omega,\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u_j(\xi,\eta)e^{i\omega\xi}d\xi \quad (j = 1, 2). \tag{33}$$

Then, (28)–(31) become

$$U_2\left(\frac{\pi\alpha}{2}\right) = (1-\rho)U_1\left(\frac{\pi\alpha}{2}\right) - sgn\omega\, f(\omega), \tag{34}$$

$$U_2'\left(\frac{\pi\alpha}{2}\right) = (1+\rho)U_1'\left(\frac{\pi\alpha}{2}\right) - \omega f(\omega), \tag{35}$$

$$U_1'(0) = 0, \tag{36}$$

$$U_2'(\pi) = 0, \tag{37}$$

where $sgn\omega$ is the sign of ω. $f(\omega)$ is defined as

$$f(\omega) = i\sqrt{2\pi}e^{-\pi(1-\frac{\alpha}{2})|\omega|}. \tag{38}$$

The Laplace equation

$$(u_j)_{xx} + (u_j)_{yy} = 0 \tag{39}$$

under the transformation (33) becomes

$$-\omega^2 U_j + U_j'' = 0. \tag{40}$$

Then,

$$U_1(\eta) = C_1(\omega)\cosh\omega\eta + E(\omega)\sinh\omega\eta, \quad 0 \le \eta \le \frac{\pi\alpha}{2}, \tag{41}$$

$$U_2(\eta) = C_2(\omega)\cosh\omega(\pi-\eta) + F(\omega)\sinh\omega(\pi-\eta), \quad \frac{\pi\alpha}{2} \le \eta \le \pi. \tag{42}$$

Using (36) and (37) imply that (41) and (42) take the form

$$U_1(\eta) = C_1(\omega)\cosh\omega\eta, \quad 0 \le \eta \le \frac{\pi\alpha}{2}, \tag{43}$$

$$U_2(\eta) = C_2(\omega)\cosh\omega(\pi-\eta), \quad \frac{\pi\alpha}{2} \le \eta \le \pi. \tag{44}$$

Substitution of (43) and (44) into (34) and (35) yields the following algebraic equations with respect to $C_1(\omega)$ and $C_2(\omega)$

$$C_1(\omega)(1-\rho)\cosh\frac{\pi\alpha}{2}\omega - C_2(\omega)\cosh\pi\left(1-\frac{\alpha}{2}\right)\omega = sgn\,\omega\, f(\omega), \tag{45}$$

$$C_1(\omega)(1+\rho)\sinh\frac{\pi\alpha}{2}\omega + C_2(\omega)\sinh\pi\left(1-\frac{\alpha}{2}\right)\omega = f(\omega). \tag{46}$$

One can find

$$C_1(\omega) = \frac{f(\omega)}{\Delta(\omega)}\left[\cosh\pi\left(1-\frac{\alpha}{2}\right)\omega + sgn\,\omega\sinh\pi\left(1-\frac{\alpha}{2}\right)\omega\right], \tag{47}$$

where

$$\Delta(\omega) = \sinh\pi\omega - \rho\sinh\pi(1-\alpha)\omega. \tag{48}$$

The coefficient $C_2(\omega)$ is expressed as

$$C_2(\omega) = \frac{f(\omega)}{\Delta(\omega)}\left[(1-\rho)\cosh\frac{\pi\alpha}{2}\omega - (1+\rho)sgn\,\omega\sinh\frac{\pi\alpha}{2}\omega\right] \tag{49}$$

Now, substitute (47)–(49) into (43) and (44) and apply the inverse Fourier transform to the resulting functions. First, consider (43)

$$u_1(\xi,\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} C_1(\omega)\cosh\omega\eta e^{-i\omega\xi}d\omega, \quad 0 \le \eta \le \frac{\pi\alpha}{2}. \tag{50}$$

$C_1(\omega)$ is seen to be an odd function. Then, (50) yields

$$u_1(\xi,\eta) = 2\int_0^{+\infty} \frac{\cosh\omega\eta\sin\omega\xi}{\sinh\pi\omega - \rho\sinh\pi\omega(1-\alpha)}d\omega, \quad \xi \ge 0, \quad 0 \le \eta \le \frac{\pi\alpha}{2}. \tag{51}$$

It is convenient to consider (51) only for $\xi \ge 0$ assuming that $u_1(\xi,\eta)$ is an odd function of ξ.

Similar arguments yield the formula

$$u_2(\xi,\eta) = 2\int_0^{+\infty} \frac{(e^{-\pi\omega} - \rho e^{-\pi(1-\alpha)\omega}) \cosh\omega(\pi-\eta)\sin\omega\xi}{\sinh\pi\omega - \rho\sinh\pi\omega(1-\alpha)} d\omega, \qquad (52)$$

$$\frac{\pi\alpha}{2} \le \eta \le \pi.$$

One can see that both integrands in (51) and (52) decay exponentially as ω tends to infinity, since $\alpha < 1$ and $\eta < \pi$. An exact asymptotic of $u_j(\xi,\eta)$ as $\xi \to \pm\infty$ is derived in the next section.

In order to present the final formulas for pressure and seepage velocity, the relation (19) between $w = \xi + \mathbf{i}\eta$ and $z = x + \mathbf{i}y$ is rewritten in real form

$$\xi = \frac{1}{2}\ln\frac{(x+1)^2+y^2}{(x-1)^2+y^2}, \qquad (53)$$

$$\tan\eta = \frac{2y}{1-x^2-y^2}, \qquad (54)$$

For instance, for positive x and y from D_i, (54) yields $\eta = \arctan\frac{2y}{1-x^2-y^2}$. The pressure has the form

$$p_i(x,y) = \frac{2}{K+1}u_1(\xi,\eta),\ (x,y) \in D_i \qquad (55)$$

$$p_e(x,y) = u_2(\xi,\eta) + \frac{2\xi}{\xi^2+(\pi-\eta)^2},\ (x,y) \in D_e.$$

where ξ and η are given by (53)–(54); $u_1(\xi,\eta)$ and $u_2(\xi,\eta)$ are calculated by (51) and (52), respectively. The seepage velocity is calculated by the pressure via formula (6).

It follows from (19) that $z = x + \mathbf{i}y$ tends to infinity if and only if w tends to $\pi\mathbf{i}$, or equivalently $\xi \to 0$, $\eta \to \pi$. We have from (52) that $u_2(0,\pi) = 0$. The term $\frac{2\xi}{\xi^2+(\pi-\eta)^2} = Re\frac{2}{w-\pi\mathbf{i}}$ from (55) has the required asymptotic as $z = x + \mathbf{i}y$ tends to infinity, since

$$Re\frac{2}{w-\pi\mathbf{i}} = Re\frac{2}{\ln\frac{z+1}{z-1}} = x + 0(1)$$

(compare (16)).

Note that the pressure is defined within an arbitrary additive constant. However, this constant was implicitly fixed in the transformation of the problem (24)–(26) to the problem (34)–(37). If we keep this arbitrary constant, a generalized δ–function arises as a result of the Fourier transformation of the constant.

3.3. Asymptotic near the angular points

This problem has already attracted some attention in different contexts. Obnosov [9] constructed radial solutions for two media of permeabilities K_e and K_i which occupy the angular domains $|\theta| > \pi\alpha$ and $|\theta| < \pi\alpha$, respectively ($\alpha < 1$) in the polar coordinates (r,θ). The complex velocities V_e and V_i were obtained in the form

$$V_e(z) = Az^{\eta-1}, \quad V_i(z) = Bz^{\eta-1}, \qquad (56)$$

where A and B are some constants, and $z = re^{i\theta}$ is a complex coordinate. Here, the real number η satisfies equation (A.1). A similar problem was discussed by Keller [5] who calculated the conductance between two highly conducting parallelograms that meet at a corner of angle $\pi\alpha$. This corresponds to the Darcy flow in the case $K_i \gg K_e$. Keller [5] derived the pressure as $p(r,\theta) = Cr^{\eta}\cos(\pi\eta\theta)$ where η satisfies

$$\tan\frac{\pi\eta\alpha}{2} = \frac{K_e}{K_i}\cot\frac{\pi\eta(1-\alpha)}{2}. \tag{57}$$

In general equations (A.1) and (57) have different roots. Therefore, special radial solutions of the corner problems were constructed in [9] and [5].

We do not know of any other analytical results about the corner problems in the literature.

Let us now go back to the flow around the fracture. For definiteness, in this subsection we consider the case $\rho > 0$. The poles of the integrand from (51) are complex solutions of the number equation

$$\sinh\pi\omega - \rho\sinh\pi(1-\alpha)\omega = 0. \tag{58}$$

This equation is completely investigated in Appendix. The roots of (58) are simple purely imaginary. Therefore,

$$G(\omega) = \frac{\cosh\eta\omega}{\sinh\pi\omega - \rho\sinh\pi(1-\alpha)\omega}, \tag{59}$$

which appears in the integrand of (51) can be expressed as

$$G(\omega) = \frac{1}{\pi}\sum_{k=0}^{\infty}\chi_k(\eta)\frac{\omega}{\omega^2+\gamma_k^2}, \tag{60}$$

where γ_k and $\chi_k(\eta)$ are given by (A.1) and (A.22), respectively.

This results can be applied to study the asymptotic behavior of the pressure and the velocity near the angular points $z = \pm 1$. Application of (60) to (51) implies

$$u_1(\xi,\eta) = \frac{2}{\pi}\sum_{k=0}^{\infty}\chi_k(\eta)\int_0^{\infty}\sin\omega\xi\frac{\omega}{\omega^2+\gamma_k^2}d\omega, \tag{61}$$

where the integrals are given by

$$\int_0^{\infty}\frac{\sin\omega\xi}{\omega}d\omega = \frac{\pi}{2}, \quad \int_0^{\infty}\frac{\omega}{\omega^2+\gamma_k^2}\sin\omega\xi d\omega = \frac{\pi}{2}e^{-\gamma_k\xi}, \quad k = 1,2,\ldots. \tag{62}$$

Therefore, applying (A.22) we obtain

$$u_1(\xi,\eta) = \frac{1}{1-\rho(1-\alpha)} + 2\sum_{k=1}^{\infty}\frac{\cos\gamma_k\eta\; e^{-\gamma_k\xi}}{\cos\pi\gamma_k - \rho(1-\alpha)\cos\pi(1-\alpha)\gamma_k}, \tag{63}$$

$$\xi > 0.$$

Recall that $u_1(\xi,\eta)$ is an odd function of ξ and that $\gamma_k \geq 0$. Formula (63) represents the full asymptotic expansion of $u_1(\xi,\eta)$ near $\xi \to +\infty$.

The asymptotic bahaviour of $p_\beta(x,y)$ for $\beta = i, e$ is expressed by (55) near the angular point $x = 1$, $y = 0$ which corresponds to $\xi = +\infty$, $\eta = 0$. Using (55), (19) and the relation

$$Re\, e^{-\gamma_k w} = \cos \eta\gamma_k\, e^{-\gamma_k \xi} \quad (w = \xi + \mathbf{i}\eta)$$

we obtain near the point $x = 1$, $y = 0$

$$p_i(x,y) = \frac{4}{K+1} \sum_{k=1}^{\infty} \frac{1}{\Delta_k'} Re \left(\frac{1-z}{1+z} \right)^{\gamma_k}, \tag{64}$$

where $z = x + \mathbf{i}y$,

$$\Delta_k' = \cos \pi\gamma_k - \rho(1-\alpha) \cos \pi(1-\alpha)\gamma_k. \tag{65}$$

The constant term in (64) is omitted. It follows from (11) that the corresponding complex potential $\varphi_i(z)$ has the form

$$\varphi_i(z) = 2 \sum_{k=1}^{\infty} \frac{1}{\Delta_k'} \left(\frac{1-z}{1+z} \right)^{\gamma_k}, \tag{66}$$

The second formula (6) in terms of complex potentials becomes

$$\mathbf{v}_i(x,y) = -\frac{2K}{K+1} \overline{\varphi_i'(z)}, \tag{67}$$

where the bar means the complex conjugation and the complex value is identified with a vector. The first–order term of (66) and (67) yield the asymptotic formula

$$\mathbf{v}_i(x,y) \sim \frac{2^{2-\gamma_1} K \gamma_1}{(K+1)\Delta_1'} \overline{(1-z)^{\gamma_1 - 1}}, \quad z \to 1. \tag{68}$$

Similar straightforward estimations of the integral (52) are possible, but they are too cumbersome. We will find first the complex potential $\varphi_e(z)$ using (66) and (13) and then the pressure p_e and the velocity $\mathbf{v}_e$. Introduce the auxiliary complex variable

$$\zeta = \frac{1+z}{1-z}. \tag{69}$$

One can consider (69) as the conformal automorphism of the upper half–plane H. Then, on the ζ–plane $H \cap D_i$ and $H \cap D_e$ become the edges $0 < arg\ \zeta < \frac{\pi\alpha}{2}$ and $\frac{\pi\alpha}{2} < arg\ \zeta < \pi$, respectively. The point $z = 1$ corresponds to the point $\zeta = \infty$. The representation (66) becomes (for brevity we use the same letter for potentials)

$$\varphi_i(\zeta) = 2 \sum_{k=1}^{\infty} \frac{1}{\Delta_k'} \zeta^{-\gamma_k}. \tag{70}$$

The $\mathbb{R}$–linear conjugation condition (13) as well as the symmetry conditions (15) keeps its form under the conformal mapping (69) (see [8])

$$\varphi_e(\zeta) = \varphi_i(\zeta) - \rho \overline{\varphi_i(\zeta)}, \quad arg\ \zeta = \frac{\pi\alpha}{2}. \tag{71}$$

Therefore, one can find $\varphi_e(z)$ in the form

$$\varphi_e(\zeta) = \sum_{k=1}^{\infty} X_k \zeta^{-\gamma_k}. \tag{72}$$

Substitution of (70) and (72) into 71 yields

$$X_k = \frac{2}{\Delta_k'}(1 - \rho\, e^{\pi \mathbf{i} \alpha \gamma_k}), \quad k = 1, 2, \dots . \tag{73}$$

Then, (11) and (72)–(73) yield the representation for $p_e(x, y)$ up to an additive constant

$$p_e(x, y) = 2\sum_{k=1}^{\infty} \frac{1}{\Delta_k'} Re\left((e^{-\pi \mathbf{i} \gamma_k} - \rho e^{-\pi \mathbf{i} \gamma_k (1-\alpha)}) \left(\frac{z-1}{z+1}\right)^{\gamma_k} \right). \tag{74}$$

Here we use the relation

$$\left(\frac{1-z}{1+z}\right)^{\gamma_k} = e^{-\pi \mathbf{i} \gamma_k} \left(\frac{z-1}{z+1}\right)^{\gamma_k}.$$

Using the first formula (6) written in the form

$$\mathbf{v}_e(x, y) = -\overline{\varphi_e'(z)} \tag{75}$$

and the principal part of (74) we obtain the following asymptotic formula

$$\mathbf{v}_e(x, y) \sim -\frac{2^{1-\gamma_1}\gamma_1}{\Delta_1'}(e^{\pi \mathbf{i} \gamma_1} - \rho e^{\pi \mathbf{i} \gamma_1 (1-\alpha)})\overline{(z-1)^{\gamma_1 - 1}}, \quad \text{as} \quad z \to 1. \tag{76}$$

4. Flow perpendicular to the lens and general flow

The same methodology as in the previous section can be applied to the transversal flow problem (13), (17)–(18). However, we apply here a simpler method based on the reduction of the anti-symmetric problem to a symmetric one.

Consider the complex potentials $\varphi_\beta^*(z)$ ($\beta = i, e$) introduced in Section 2 corresponding to the flow perpendicular to the lens. They satisfy the $\mathbb{R}$–linear problem

$$\varphi_e^*(t) = \varphi_i^*(t) - \rho\overline{\varphi_i^*(t)}, \quad t \in \Gamma_1, \tag{77}$$

with the conditions (17)–(18). Introduce the auxiliary complex potentials

$$\phi_\beta(z) = \mathbf{i}\varphi_\beta^*(z), \quad z \in D_\beta. \tag{78}$$

It is easily seen that they satisfy the following problem

$$\phi_e(t) = \phi_i(t) + \rho\overline{\phi_i(t)}, \quad t \in \Gamma_1, \tag{79}$$

$$\phi_i(z) = \overline{\phi_i(\overline{z})}, \quad \phi_e(z) = \overline{\phi_e(\overline{z})}, \tag{80}$$

$$\phi_e(z) \sim z, \quad \text{as} \quad z \to \infty. \tag{81}$$

The problem (79)–(81) differs of the symmetric problem (13), (15)–(16) only by the sign of ρ. Therefore, in order to obtain formulas for pressure and velocity, we have to take the symmetric solution from the previous section, to replace ρ by $(-\rho)$ and to perform the transformation inverse to (78). Let us note that by replacing ρ by $-\rho$, equation (58) is replaced by

$$\sinh \pi\omega + \rho \sinh \pi(1-\alpha)\omega = 0, \tag{82}$$

which is also studied in Appendix.

Ultimately, we obtain

$$p_i^*(x,y) = \frac{2}{K+1} u_1^*(\xi,\eta),\ (x,y) \in D_i, \tag{83}$$

$$p_e^*(x,y) = u_2^*(\xi,\eta) + \frac{2(\pi-\eta)}{\xi^2+(\pi-\eta)^2},\ (x,y) \in D_e.$$

where ξ and η are given by (53)–(54). Let us explain how to obtain $u_1^*(\xi,\eta)$ and $u_2^*(\xi,\eta)$ from $u_1(\xi,\eta)$ and $u_2(\xi,\eta)$ given by (51) and (52). First, we note that the complex potentials $\Phi_\beta(w) = \phi_\beta(z)$, where w and z are related by (19), also satisfy the condition $\Phi_\beta(w) = \overline{\Phi_\beta(\overline{w})}$. The term from (53) which contains the variable ξ and η generates an analytic function

$$\sin\omega w = \cosh\omega\eta \sin\omega\xi + \mathbf{i}\sinh\omega\eta\cos\omega\xi, \tag{84}$$

where $w = \xi + \mathbf{i}\eta$. It follows from (78) that

$$Re\ \Phi_\beta^*(w) = Im\ \Phi_\beta(w). \tag{85}$$

Therefore, in order to obtain $u_1^*(\xi,\eta)$ from $u_1(\xi,\eta)$ according to (85), we have to replace $Re\ \Phi_\beta(w)$ by $Im\ \Phi_\beta(w)$ and to replace ρ by $-\rho$. Together with (84) this transformation yields

$$u_1^*(\xi,\eta) = 2\int_0^{+\infty} \frac{\sinh\omega\eta\cos\omega\xi}{\sinh\pi\omega + \rho\sinh\pi\omega(1-\alpha)} d\omega, \tag{86}$$

$$\xi \geq 0, \quad 0 \leq \eta \leq \frac{\pi\alpha}{2}.$$

Similar arguments yield

$$u_2^*(\xi,\eta) = 2\int_0^{+\infty} \frac{(e^{-\pi\omega} + \rho e^{-\pi(1-\alpha)\omega})\sinh\omega(\pi-\eta)\cos\omega\xi}{\sinh\pi\omega + \rho\sinh\pi\omega(1-\alpha)} d\omega, \tag{87}$$

$$\frac{\pi\alpha}{2} \leq \eta \leq \pi.$$

We now study the asymptotic behaviour of the solution near the point $x = 1$, $y = 0$ which in polar coordinates corresponds to the point $r = 0$. For definiteness, we consider the case $\rho > 0$. Then up to an additive constant (compare with (64))

$$p_i^*(x,y) = \frac{4}{K+1} \sum_{k=1}^{\infty} \frac{1}{\Delta_k''} Im \left(\frac{1-z}{1+z}\right)^{\delta_k}, \tag{88}$$

where $\pm\mathbf{i}\delta_k$ are the roots of equation (82),

$$\Delta_k'' = \cos\pi\delta_1 + \rho(1-\alpha)\cos\pi(1-\alpha)\delta_1.$$

Along similar lines we have

$$p_e^*(x,y) = 2\sum_{k=1}^{\infty} \frac{1}{\Delta_k''} Im \left((e^{-\pi\mathbf{i}\delta_k} - \rho e^{-\pi\mathbf{i}\delta_k(1-\alpha)}) \left(\frac{z-1}{z+1}\right)^{\delta_k} \right). \tag{89}$$

The velocity $\mathbf{v}^*(x,y)$ has the following asymptotic behaviour near $z=1$ (compare with (68))

$$\mathbf{v}_i^*(x,y) \sim -\mathbf{i}\frac{2^{2-\delta_1}K\delta_1}{(K+1)\Delta_1''}\overline{(1-z)^{\delta_1-1}}, \tag{90}$$

$$\mathbf{v}_e^*(x,y) \sim -\mathbf{i}\frac{2^{1-\delta_1}\delta_1}{\Delta_1''}(e^{\pi\mathbf{i}\delta_1} - \rho e^{\pi\mathbf{i}\delta_1(1-\alpha)})\overline{(z-1)^{\delta_1-1}}. \tag{91}$$

In contrast with flow parallel to the lens, the velocities $\mathbf{v}_\beta$ are equal to zero at $z=1$ because of (A.5).

Now, consider the general case (9). Then,

$$p(x,y) = \frac{2}{K+1}[p_{,1\infty}u_1(\xi,\eta) + p_{,2\infty}u_1^*(\xi,\eta)], \quad (x,y)\in D_i, \tag{92}$$

where $u_1(\xi,\eta)$ and $u_1^*(\xi,\eta)$ have the form (51) and (86), respectively; (x,y) and (ξ,η) are related by (53)–(54). We also have

$$p_e(x,y) = p_{,1\infty}u_2(\xi,\eta) + p_{,2\infty}u_2^*(\xi,\eta), \quad (x,y)\in D_e, \tag{93}$$

where $u_2(\xi,\eta)$ and $u_2^*(\xi,\eta)$ are calculated by (52) and (87), respectively.

In order to study the asymptotic behaviour near the point $x=1$, $y=0$, compare the asymptotics (76) and (91). It follows from Lemma 1 that $0<\gamma_1<1<\delta_1$. Hence, the main asymptotic term of the velocity in the lens becomes

$$\mathbf{v}_i(x,y) \sim p_{,1\infty}\frac{2^{2-\gamma_1}K\gamma_1}{(K+1)\Delta_1'}\overline{(1-z)^{\gamma_1-1}}, \tag{94}$$

if only $p_{,1\infty}\neq 0$. In the case of the external flow perpendicular to the lens ($p_{,1\infty}=0$) in accordance with (90) and (92), we have

$$\mathbf{v}_i(r,\theta) \sim -p_{,2\infty}\mathbf{i}\frac{2^{2-\delta_1}K\delta_1}{(K+1)\Delta_1''}\overline{(1-z)^{\delta_1-1}}. \tag{95}$$

5. Discussion and conclusion

For a check, let us compare the analytical solution to some numeric solution. This is more precise for $\alpha=\frac{1}{2}$ since the two arcs meet with a right angle. Then, equation (A.1) is easily solved. For positive ρ we have

$$\gamma_1 = \frac{2}{\pi}\arccos\frac{\rho}{2},\ \gamma_2 = 4-\frac{2}{\pi}\arccos\frac{\rho}{2},\ \gamma_3=\gamma_1+4,\ \gamma_4=\gamma_2+4,\ldots, \tag{96}$$

$$\delta_1 = 2-\frac{2}{\pi}\arccos\frac{\rho}{2},\ \delta_2 = 2+\frac{2}{\pi}\arccos\frac{\rho}{2},\ \delta_3=\delta_1+4,\ \delta_4=\delta_2+4,\ldots. \tag{97}$$

Consider an example with $K=12$. Calculations were performed on a square grid by a finite volume technique. The boundary conditions are different since the medium is spatially periodic. The width of the lens is about 20% the size of the cell. The cell was discretized into N_c^2 squares of permeabilities 1 and K. Computations of $p_\beta(x,y)$ $(\beta=i,e)$ are presented in Figure 3, where $p_i(x,y)$ and $p_e(x,y)$ are calculated by (64) and (74), respectively.

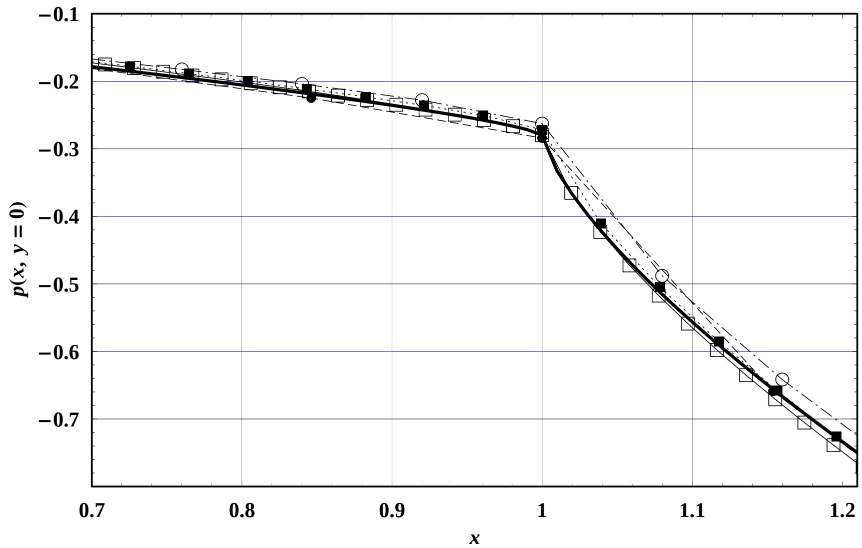

Figure 3. Data are for: present calculations (solid line); numerical calculations (broken lines): $N_c = 64$ (full circle), 128 (∘), 256 (full square), 512 (empty square).

In the present paper we have solved in closed form the problem of the flow around a permeable lens immersed in an infinite porous medium of different permeability.

Let $K > 1$. Then, for flow parallel to the lens, the pressure is given by (55), (51)–(52), where ξ and η are given by (53)–(54). The seepage velocity is derived from pressure via formula (6). The singularity of the velocity near the angular point of the lens $x = 1$, $y = 0$ are described by (68) and (76). It is worth noting that the velocity has a power singularity at the angular points expressed by $(z-1)^{\gamma_1}$, where γ_1 is the minimal positive root of equation (A.1) satisfying inequality (A.3).

Let now $K < 1$. Then, for flow parallel to the lens, the velocity vanishes at the angular points. For flow perpendicular to the lens, the velocity has the singularity expressed by $(z-1)^{\gamma_1}$.

For general external flow not parallel to the axes, the velocity has a singularity of the form $(z-1)^{\gamma_1}$ for any K. Therefore, the two special cases discussed above when the velocity has no any singularity at the angular points, are very sensitive to the direction of the external field.

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Appendix

In the present section, we investigate the asymptotic behavior of the integral (51). The poles of the integrand of (51) are complex solutions of the number equation (58). We shall first look for properties of the purely imaginary roots $\omega = \mathbf{i}\eta$ of equation (58).

Lemma 1 *Let $0 < \rho < 1$, $0 < \alpha < 1$. All real roots η of equation*

$$\sin \pi\eta - \rho \sin \pi(1-\alpha)\eta = 0 \tag{A.1}$$

are simple and can be arranged as follows $\gamma_0 = 0, \pm\gamma_1, \pm\gamma_2 \ldots$, where $0 < \gamma_1 < \gamma_2 < \ldots$. Moreover,

$$k - \nu < \gamma_k < k + \nu, \quad k = 1, 2, \ldots, \tag{A.2}$$

where $\nu = \frac{1}{\pi} \arcsin \rho$. The root γ_1 satisfies the inequality

$$1 - \nu < \gamma_1 < 1. \tag{A.3}$$

Let $-1 < \rho < 0$, $0 < \alpha < 1$. All real roots η of equation (A.1) are simple and can be arranged as follows $\delta_0 = 0, \pm\delta_1, \pm\delta_2 \ldots$, where $0 < \delta_1 < \delta_2 < \ldots$. Moreover,

$$k - \nu < \delta_k < k + \nu, \quad k = 1, 2, \ldots, \tag{A.4}$$

The root δ_1 satisfies the inequality

$$1 < \delta_1 < 1 + \nu. \tag{A.5}$$

Proof. For definiteness, we take $0 < \rho < 1$. Equation (A.1) with positive roots is equivalent to the following set of equations

$$\eta = \frac{1}{\pi} \arcsin[\rho \sin \pi(1-\alpha)\eta] + 2m, \tag{A.6}$$

$$\eta = -\frac{1}{\pi} \arcsin[\rho \sin \pi(1-\alpha)\eta] + 2m - 1, \quad m = 1, 2, \ldots \tag{A.7}$$

For each fixed m, the method of successive approximations can be applied to (A.6) and to (A.7), since

$$\left| \frac{\partial}{\partial\eta} \frac{1}{\pi} \arcsin[\rho \sin \pi(1-\alpha)\eta] \right| \le (1-\alpha)\rho < 1. \tag{A.8}$$

In particular, equation (A.6) with $m = 0$ is omitted, because it has only a trivial solution $\gamma_0 = 0$. Therefore, each equation (A.6)–(A.7) has a unique solution. The root γ_k is obtained from (A.6) if $k = 2m$ and from (A.7) if $k = 2m - 1$.

Consider the root γ_1 satisfying (A.7) for $m = 1$. We have

$$\arcsin[\rho \sin \pi(1-\alpha)\eta] > 0,$$

and

$$\frac{1}{\pi} |\arcsin[\rho \sin \pi(1-\alpha)\eta]| \le \nu < \frac{1}{2}. \tag{A.9}$$

since $\rho > 0$, $\gamma_1 > 0$ and $\alpha < 1$. This yields (A.3). Along similar lines, the relation (A.2) follows from (A.6), (A.7) and the inequality (A.9).

Let us check that all roots of (A.1) are simple. If it is not true for some γ_k, differentiation yields

$$\cos \pi\eta - \rho(1-\alpha)\cos \pi(1-\alpha)\eta = 0. \tag{A.10}$$

Then, (A.1) and (A.10) imply

$$1 = \rho^2[\sin^2 \pi(1-\alpha) + (1-\alpha)^2 \cos^2 \pi(1-\alpha)\eta]. \tag{A.11}$$

The right hand side of (A.11) is less than ρ^2 which is less than unity. Since this is contradictory, the lemma is proved.

Lemma 2 *Let $0 < |\rho| < 1$, $0 < \alpha < 1$. All complex roots of equation (58) are simple and have the form $\omega = \pm i\gamma_k$ $(k = 0, 1, \ldots)$, where γ_k are real roots of equation (A.1) for positive ρ, and $\omega = \pm i\delta_k$ for negative ρ.*

Proof. For definiteness, consider the case $0 < \rho < 1$. First, recall Rouché's theorem of classical complex analysis. Take two functions $f(z)$ and $g(z)$ analytic in a simply connected domain $\overline{\Omega} = \Omega \cup \partial\Omega$. Let $|f(z)| > |g(z)|$ for all z on $\partial\Omega$. Then, $f(z)$ and $f(z) + g(z)$ have the same numbers of roots in Ω. We apply Rouché's theorem to the functions

$$f(\omega) = \frac{\sinh \pi\omega}{\omega}, \text{ and } g(\omega) = -\rho\frac{\sinh \pi(1-\alpha)\omega}{\omega} \tag{A.12}$$

in the rectangles $\Omega_k = \{\omega = \xi + \mathbf{i}\eta \in \mathbb{C} : -A < \xi < A, 0 < \eta < B_k\}$, where A is a sufficiently large positive number; the sequence $B_k \in (k, k + \nu)$ is chosen in such a way that

$$|\sin \pi B_k| > \rho|\sin \pi(1-\alpha)B_k|. \tag{A.13}$$

Let us demonstrate that such a sequence B_k exists. Let γ_k and δ_k be the roots of functions $\sin \pi\eta - \rho \sin \pi(1-\alpha)\eta$ and $\sin \pi\eta + \rho \sin \pi(1-\alpha)\eta$, respectively. For definiteness, consider an irrational α. According to Lemma 1, these roots are simple and lie in the segment $(k - \nu, k + \nu)$. Moreover, $\gamma_k \neq \delta_k$, since $\rho \neq 0$ and α is irrational. Therefore, the segment $(k - \nu, k + \nu)$ is divided into three small segments by the points γ_k and δ_k. The function $F(\eta) = \sin^2 \pi\eta - \rho^2 \sin^2 \pi(1-\alpha)\eta$ is positive for small $\eta > 0$, and it changes its sign at $\eta = \gamma_k$ and $\eta = \delta_k$. For definiteness, let $\gamma_k < \delta_k$. Then, $F(\eta)$ is negative for $\eta \in (\gamma_k, \delta_k)$ and positive for $\eta \in (k - \nu, \gamma_k) \cup (\delta_k, k + \nu)$. Therefore, such a $B_k \in (k, k + \nu)$ exists such that $F(B_k) > 0$. Therefore, (A.13) is fulfilled.

We must now check that $|f(\omega)| > |g(\omega)|$ on $\partial\Omega_k$. The boundary of Ω_k consists of four segments. However, since $f(\omega)$ and $g(\omega)$ are even functions, it is sufficient to check the inequality only for

i) $0 \le \xi \le A$, $\eta = 0$,
ii) $\xi = A$, $0 \le \eta \le B_k$,
iii) $0 \le \xi \le A$, $\eta = B_k$.

i) In order to prove that

$$\sin \pi\xi > \rho \sin \pi(1-\alpha)\xi \quad \text{for} \quad 0 < \xi \le A, \tag{A.14}$$

write (A.14) in the form

$$\rho e^{\pi\xi} + (1-\rho)e^{\pi\xi} + \rho e^{-\pi(1-\alpha)\xi} > \rho e^{\pi(1-\alpha)\xi} + \rho e^{-\pi\xi} + (1-\rho)e^{-\pi\xi}. \tag{A.15}$$

The latter equality is fulfilled, because $e^{\pi\xi} > e^{\pi(1-\alpha)\xi}$, $e^{\pi\xi} > e^{-\pi\xi}$ and $e^{-\pi(1-\alpha)\xi} > e^{-\pi\xi}$ for $\xi > 0$. Therefore, (A.14) is true.

ii) In order to prove that

$$|\sinh \pi(A + \mathbf{i}\eta)| > \rho|\sinh \pi(1-\alpha)(A+\mathbf{i}\eta)| \quad \text{for} \quad 0 \le \eta < B_k, \tag{A.16}$$

use the formula

$$|\sinh(x + \mathbf{i}y)|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y.$$

(A.16) implies

$$\begin{aligned} &\sinh^2 \pi A \cos^2 \pi\eta + \cosh^2 \pi A \sin^2 \pi\eta > \\ &\rho^2[\sinh^2 \pi(1-\alpha)A \cos^2 \pi(1-\alpha)\eta + \cosh^2 \pi(1-\alpha)A \sin^2 \pi(1-\alpha)\eta]. \end{aligned} \tag{A.17}$$

Using the relations $\cos^2 y = 1 - \sin^2 y$ and $\sinh^2 x = \cosh^2 x - 1$, (A.17) is rewritten as

$$\begin{aligned} &\sinh^2 \pi A + \sin^2 \pi\eta > \\ &\rho^2(\sinh^2 \pi(1-\alpha)A + \sin^2 \pi(1-\alpha)\eta) \quad \text{for} \quad 0 \le \eta < B_k. \end{aligned} \tag{A.18}$$

For sufficiently large A, since $\rho < 1$ and $\alpha < 1$, we have

$$\sinh^2 \pi A > \rho^2(\sinh^2 \pi(1-\alpha)A + 1), \tag{A.19}$$

(A.19) implies (A.18) and hence (A.16).

iii) In order to prove that

$$|\sinh \pi(\xi + \mathbf{i}B_k)| > \rho|\sinh \pi(1-\alpha)(\xi + \mathbf{i}B_k)| \quad \text{for} \quad 0 \le \xi \le A \tag{A.20}$$

as in the previous case, (A.20) is reduced to

$$\sinh^2 \pi\xi + \sin^2 \pi B_k > \rho^2(\sinh^2 \pi(1-\alpha)\xi + \sin^2 \pi(1-\alpha)B_k) \quad \text{for} \quad 0 \le \xi \le A.$$

The latter inequality follows from $\sinh \pi\xi \ge \sinh^2 \pi(1-\alpha)\xi$ and (A.13).

Therefore, Rouché's theorem can be applied. All roots of $f(\omega)$ in Ω_k have the form $\omega = im$ $(m = 1, 2, \ldots, k)$, because $\sinh \pi(\xi + \mathbf{i}\eta) = 0$ if and only if $\sinh \pi\xi = 0$ and $\sin \pi\eta = 0$. Then, the function $f(\omega) + g(\omega)$ has also k roots in Ω_k. Exactly k roots of $f(i\eta) + g(i\eta)$ are described in Lemma 1. Hence, no other root of $f(\omega) + g(\omega)$ can be found in Ω_k. Finally, k is increased up to infinity.

This proves the lemma.

Lemma 3 *Let γ_k be the roots of equation (A.1). Then,*

$$G(\omega) = \frac{\cosh \eta\omega}{\sinh \pi\omega - \rho \sinh \pi(1-\alpha)\omega} = \sum_{k=0}^{\infty} \chi_k(\eta) \frac{\omega}{\omega^2 + \gamma_k^2}, \tag{A.21}$$

where

$$\chi_0(\eta) = \frac{1}{\pi(1-\rho(1-\alpha))}, \tag{A.22}$$

$$\chi_k(\eta) = \frac{2\cos\eta\gamma_k}{\pi[\cos\pi\gamma_k - \rho(1-\alpha)\cos\pi(1-\alpha)\gamma_k]}, \quad k = 1, 2, \dots$$

The series in (A.21) converges absolutely and uniformly for $-\infty < \omega < +\infty$.

Proof. In the complex plane, consider the representation of the meromorphic function following from the Mittag–Leffler theorem [7]

$$G(\omega) = \frac{1}{2}\sum_{k=0}^{\infty}\chi_k(\eta)\left(\frac{1}{\omega - \mathbf{i}\gamma_k} + \frac{1}{\omega + \mathbf{i}\gamma_k}\right) \tag{A.23}$$

According to Lemma 2, the function $G(\omega)$ for positive ρ has poles only at the points $\omega = \pm\mathbf{i}\gamma_k$. All these poles are first order. Calculating the residue of $G(\omega)$ at $\omega = \pm\mathbf{i}\gamma_k$, we obtain (A.21)–(A.22).

In order to verify the absolute and uniform convergence of the series from (A.21), note that $\gamma_k \sim k$, as $k \to \infty$ in accordance with Lemma 1. Moreover, the denominators of $\chi_k(\eta)$ are uniformly bounded, i.e.,

$$|\cos\pi\gamma_k - \rho(1-\alpha)\cos\pi(1-\alpha)\gamma_k| \geq c_0 > 0, \quad k = 0, 1, \dots \tag{A.24}$$

We prove (A.24) by contradiction. Assume the existence of a subsequence of the roots γ_{k_m} such that $\varepsilon_m := \cos\pi\gamma_{k_m} - \rho(1-\alpha)\cos\pi(1-\alpha)\gamma_{k_m}$ tends to zero as $m \to \infty$. Then,

$$\cos^2\pi\gamma_{k_m} = \rho^2(1-\alpha)^2\cos^2\pi(1-\alpha)\gamma_{k_m} + \varepsilon'_m, \tag{A.25}$$

where $\varepsilon'_m = \varepsilon_m[2\rho(1-\alpha)\cos\pi(1-\alpha)\gamma_{k_m} + \varepsilon_m]$. ε'_m also tends to zero as $m \to \infty$. It follows from (A.1) that

$$\sin^2\pi\gamma_{k_m} = \rho^2\sin^2\pi(1-\alpha)\gamma_{k_m}. \tag{A.26}$$

Addition of (A.25) and (A.26) yields the contradictory equality

$$1 = \rho^2[\sin^2\pi(1-\alpha)\gamma_{k_m} + \rho^2(1-\alpha)^2\cos^2\pi(1-\alpha)\gamma_{k_m}] + \varepsilon'_m, \tag{A.27}$$

since the right hand part of (A.27) is less than $\rho^2 + \varepsilon'_m$ which is less than unity.

The lemma is proved.

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