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Boundary value problems for the Laplace equation, the Poincare series, the method of Schwarz and composite materials

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ABSTRACT

The generalized method of Schwarz allows us to construct the solution of the Dirichlet or Neumann problems for a domain with multi-component boundary in the form of the series. If the boundary of the domain consists of the mutually disjoint spheres then for the Dirichlet problem we obtain the Poincare series. In the previous works the absolute and uniform convergence of these series has been discussed. It has been note that such a series can be absolutely divergent. It depends on the domain. In the present paper we separate the absolute and uniform convergence, and prove the uniform convergence of the series for arbitrary domain. The problem considered is applied to composite materials.

1. INTRODUCTION

The classical *method of images* [1, 2] can be used to solve the classical boundary value problems of Dirichlet or Neumann involving circular or spherical boundaries. It is possible to construct a formal solution of the Dirichlet problem which boundary consists of finite number of disjoint spheres. This solution is written in the form of series. Golusin [3] showed that if the spheres are sufficiently far removed from one another, then this series is absolutely convergent. If the number of spheres is equal to two, then convergence holds too. In the present paper uniform convergence has been proved for each spherical domain.

Golusin [3], Mityushev [4 - 6] considered the method of images as a special case of the *method of functional equations*. In the spatial case for the Dirichlet problem the both methods lead to the same series. Mityushev [4 - 6] modified the method of functional equations in the plane to get a convergent series. The last series is related to the *Poincare* θ*-series*. Akaza&Inouue [7, 8] constructed an example of the absolutely divergent Poincare θ-series of second order. Mityushev [5, 15] proved the uniform convergence of the Poincare θ-series of second order in the plane. It follows from Sec.4 of the present paper the Poincare series converges uniformly for each spherical domain in the space.

The *generalized method of Schwarz* has been studied in [3, 9 - 11]. For a spherical domain this method coincides with the method of functional equations. Let us note that "usual" alternating method of Schwarz always converges [12]. However, there is the same question of convergence for the generalized method. This question has been solved in [6] for the Dirichlet problem. In Sec.3 of the present paper this question has been solved for the Neumann problem.

Let us consider the mutually disjoint bounded simple domains D_k with Lyapunov's boundary ∂D_k ($k = 0,1,...,n$). Let D_k^- be the complement of $\overline{D_k}$ in the space R^3, $D:=\left(R^3 \setminus \cup_{k=0}^{n} \overline{D_k}\right) \cup \{\infty\}$. The curve ∂D is orientated in the positive direction. We assume that $\partial D := -\cup_{k=0}^{n} \partial D_k$. Let $x = (x_1, x_2, x_3) \in R^3, |x| = \left(x_1^2, x_2^2, x_3^2\right)^{1/2}$. In Sec.2 we consider the *Dirichlet* problem

$$u(t) = f(t`), \quad lim_{|x| \to \infty} u(x) =: u(\infty) = 0. \tag{1}$$

We shall write *x*, when we consider a relation in a domain and *t* - in a surface. We assume that the given function $f \in C^1(\partial D)$, i.e. *f* is continuously differentiable in ∂D. Then the unknown function $u \in C^1(\overline{D})$. In Sec. 3 we solve the *Neumann* problem

$$\frac{\partial u}{\partial n}(t) = g(t), \; t \in \partial D, \; u(\infty) = 0, \tag{2}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative. As consequence in Sec.4 we show uniform convergence of the Poincare θ-series for each spherical domain.

We shall use the classical potentials

$$Q_k \mu(x) := \frac{1}{4\pi} \iint_{\partial D_k} \frac{\mu(y)}{|y - x|} ds_y, \; P_k \mu(x) := \frac{1}{4\pi} \iint_{\partial D_k} \mu(y) \frac{\partial}{\partial n} \left(\frac{1}{|y - x|} \right) ds_y$$

for which

$$P_k^+ \mu(t) - P_k^- \mu(t) = \mu(t), \quad \frac{\partial P_k^+ \mu(t)}{\partial n} = \frac{\partial P_k^- \mu(t)}{\partial n},$$

$$Q_k^+ \mu(t) = Q_k^- \mu(t), \quad \frac{\partial Q_k^+ \mu(t)}{\partial n} - \frac{\partial Q_k^- \mu(t)}{\partial n} = \mu(t), y \in \partial D_k.$$

Here $\mu \in C^1(\partial D_k,)$ $\quad P_k^+ \mu(t) := lim_{x \to t, x \in D} P_k \mu(x)$, $\quad P_k^- \mu(t) := lim_{x \to t, x \in D_k} P_k \mu(x)$, $Q_k^+ \mu(t)$ and $Q_k^- \mu(t)$ are defined by the same way.

Let us consider the Banach space **B** which consists of functions harmonic in $\cup_{k=0}^{n} D_k$ and continuous in $\cup_{k=0}^{n} \overline{D}_k$. The norm $\|f\| := max_k \, max_{\partial D_k} |f(t)|$. Harnack's theorem implies that convergence in **B** means uniform convergence. Introduce the spaces $h^-(G) := \left\{ u \in C^1(\overline{G}) : \Delta u = 0 \; in \; G, u(\infty) = 0 \right\}$, when the domain *G* contains the infinite point, and $h^+(G) := \left\{ u \in C^1(\overline{G}) : \Delta u = 0 \; in \; G \right\}$, when *G* is bounded.

2. Dirichlet problem

Let us consider the integral operator S_k which is defined by the following Dirichlet problem

$$S_k f(t) = f(t), \quad t \in \partial D_k,$$

with respect to the function $S_k f \in h^-(D_k)$. Here $f \in C^1(\partial D_k)$.

Theorem 1 [6]. *The Dirichlet problem (1) has the unique solution*

$$u(x) = \sum_{k=0}^{n} S_k f(x) - \sum_{k=0}^{n} \sum_{k_1 \neq k} S_k S_{k_1} f(x) + \sum_{k=0}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} S_k S_{k_1} S_{k_2} f(x) - \ldots, \tag{3}$$

where $k = 0,1,\ldots,n; k_1 = 0,1,\ldots,n; \; k_1 \neq k$ *in the sum* $\sum_{k=0}^{n} \sum_{k_1 \neq k}$. *The series (3) converges uniformly in* $\overline{D}$ *and corresponds to the generalized method of Schwarz.*

In the present section Theorem 1 is applied to the case of spherical domains D, when $D_k := \left\{ x \in R^3, \quad |x - a_k| = r_k \right\} \left(k = 0,1,\ldots,n \right)$. In this case the integral operator $S_k : h^+ \left(D_k \right) \to h^- \left(D_k^- \right)$ becomes the shift operator

$$S\ w(x) = \frac{r_k}{|x - a_k|} w\left(x_k^* \right) \tag{4}$$

for each $w \in h^+ \left(D_k \right)$. Here $x_k^* := \dfrac{r_k{}^2}{|x - a_k|^2} \left(x - a_k \right) + a_k$ is the inversion with respect to the sphere ∂D_k. The representation (4) allows us to simplify the series (3) and obtain the exact solution of the Dirichlet problem (1).

For a spherical domain the function $f(x)$ takes the form

$$f(x) = \frac{1}{4\pi r_k} \iint_{\partial D_k} f(t) \frac{r_k^2 - |x - a_k|^2}{|t - x|^3} ds_t .$$

Let us denote the sequence of the inversion with respect to the spheres with numbers $k_1, k_2, \ldots, k_m$ by $x^*_{k_m k_{m-1} \ldots k_1} := \left(x^*_{k_{m-1} \ldots k_1} \right)^*_{k_m}$. There are no equal neighbor numbers in the sequence $k_1, k_2, \ldots, k_m$. Using (4) we rewrite (3) in the form

$$u(x) = \sum_{k=0}^{n} f\left(x_k^* \right) - \sum_{k=0}^{n} \sum_{k_1 \neq k} \frac{r_k}{|x - a_k|} \frac{r_{k_1}}{\left| x_k^* - a_{k_1} \right|} f\left(x^*_{k_1 k} \right) + \ldots \quad . \tag{5}$$

The last series converges uniformly in $\overline{D}$. Let us note that the series (5) involves rational transformations and doesn't contain integral operators. Moreover, if the boundary date $f(t)$ is polynomial, then $f(x)$ can be readily calculated by the algorithm of Axler & Ramey [13].

3. Neumann problem

Following [6] and Sec.2 we consider the operator $T_k : h^+ \left(D_k \right) \to h^- \left(D_k \right)$ which is defined by the following way. For $u_k \in h^+ \left(D_k \right)$ calculate $\partial u_k / \partial n$ on ∂D_k and solve the Neumann problem $\partial v_k / \partial n = \partial u_k / \partial n$ on ∂D_k with respect to $v_k \in h^- \left(D_k^- \right)$. Thus we construct the operator $T_k u_k(x) := v_k(x), x \in D_k^-$. It follows from the definition that the operator satisfies the identity

$$\frac{\partial}{\partial n} \left(T_k u_k \right)(t) = \frac{\partial u_k}{\partial n}, \quad t \in \partial D_k \quad \text{for each} \quad u_k \in h^+ \left(D_k \right). \tag{6}$$

The Neumann problems is reduced to the system of integral equations

$$u_k(x) = -\sum_{m \neq k} T_m u_m(x) - f(x), \quad x \in D_k, \quad k = 0,1,\ldots,n, \tag{7}$$

with respect to $u_k(x)$ and

$$u(x) = -\sum_{m=0}^{n} T_m u_m(x) - h(x), \quad x \in \overline{D}.$$

Lemma 1. *The system (7) has the unique solution*

$$u_k(x) = -f(x) + \sum_{k_1 \neq k} T_{k_1} f(x) - \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} T_{k_1} T_{k_2} f(x) + \ldots ,)$$

The last series converges uniformly in $\overline{D_k}$.

Theorem 2. *The Neumann problem (2) has the unique solution*

$$u(x) = -h(x) + \sum_{k=0}^{n} T_k f(x) - \sum_{k=0}^{n} \sum_{k_1 \neq k} T_k T_{k_1} f(x) + \ldots, \qquad (8)$$

The last series converges uniformly in $\overline{D}$.

The Dirichlet problem for a spherical domain has been solved in Sec.2 because the integral operator S_k has been written as the shift operator (4). We cannot solve the Neumann problem for a spherical domain in terms of non-integral operators, because we cannot write the operator T_k in a simple form. It is related to the complicated method of images for the Neumann problem. Using asymptotic expansions for the imaging rule, Poladian [2] overcame this obstacle for two spheres. Applying Poladian's formalism to T_k it is possible to obtain a simple asymptotic representation for the series (8).

4. Poincare θ–series

The transformations $x^*_{k_m k_{m-1} \ldots k_1}$ having been introduced in Sec.2 generate some Schottky group **K** [7, 8]. The number m is called the level of transformation. Let us fix the transformations $\gamma_j(x) = x^*_{k_m k_{m-1} \ldots k_1}$ (j = 0,1,2,...) in order of increasing level, i.e. $\gamma_0(x) = x, \gamma_1(x) = x_1^*, \ldots$. There holds the relation

$$\left|\gamma_j'(x)\right|^{1/2} = r_k / \left|x - a_k\right|, \quad k = 0,1,\ldots,n-1; \quad j = k+1,$$

where $\left|\gamma_j'(x)\right|$ is the measure of local stretching at the point x. The elements $\gamma_j(x)$ (j = 0,1, ..., n) are generators of the group **K**. It can be shown that

$$\left|\gamma_j'(x)\right|^{1/2} = \left(r_{k_1} / \left|x - a_{k_1}\right|\right)\left(r_{k_2} / \left|x^*_{k_1} - a_{k_2}\right|\right)\ldots\left(r_{k_m} / \left|x^*_{k_{m-1}} - a_{k_m}\right|\right)$$

for $\gamma_j(x) = x^*_{k_m k_{m-1} \ldots k_1}$.

Definition. *Let* $W = \{w_1, w_2, \ldots w_n\}$ *be a set of points belonging to D. Let f(x) be a function harmonic in* $\left(R^3 \cup \{\infty\}\right) - W$. *The series*

$$\theta(x) := \sum_{j=0}^{\infty} f\left[\gamma_j(x)\right]\left|\gamma_j'(x)\right|^{1/2} = \sum_{\gamma_j \in K} f\left[\gamma_j(x)\right]\left|\gamma_j'(x)\right|^{1/2} \qquad (9)$$

is called the Poincare θ*-series according to the group* ***K***.

Put $G := D - \overline{V}$, where $V := \left\{x \in R^3, \left|x - w_k\right| < \varepsilon, \quad k = 1,2,\ldots,m\right\}$ be spheres of the sufficiently small radii ε.

Theorem 3. *The series (9) converges uniformly in* $R^3 \cap (D - W)$ *for each group K and each set* $W \subset D$.

The proof of the theorem is based on the uniform convergence of the series (9) because the series (5) and (9) are related by the equality $f(x) + u(x) = \theta(x)$.

5. Composite materials

A problem of great theoretical and practical interest is that of calculating the effective transport properties of periodic composite materials. Following [14, 15] we consider a lattice **Q** which is defined by three fundamental translation vectors $\boldsymbol{\omega}_i$ $(i=1,2,3)$ in the space R^3. The zero cell Q_0, the basis of **Q**, is the set $\left\{x=\sum_{i=1}^{3} t_i \boldsymbol{\omega}_i, \quad -1/2 < t_i < 1/2\right\}$. Let the volume holds $|Q_0|=1$. Let $\{e_k\}_{k=1}^{\infty}$ be a linearly ordered set of the vectors $\sum_{i=1}^{3} m_i t_i$, where m_i are integer, e_0 = **0**. The lattice **Q** consists of the cells $Q_k = Q_0 + e_k := \left\{x \in R^3 : x - e_k \in Q_0\right\}$.

Let us consider a ball $D_1 := \left\{x \in R^3 : |x| < r\right\}$ in the zero cell Q_0. Let $D := Q_0 - \overline{D_1}$. We study the conductivity of the periodic composite materials, when the domains $D + e_k$ and $D_1 + e_k$ are occupied by materials of conductivity $\lambda = 1$ and λ_1, respectively. We find the potentials $u(x)$ and $u_1(x)$ to be harmonic in $D + e_k$ and $D_1 + e_k$ with the boundary conditions:

$$u = u_1, \quad \frac{\partial u}{\partial n} = \lambda_1 \frac{\partial u_1}{\partial n} \quad \text{on the sphere } |t| = r. \tag{10}$$

Moreover, the function $u(x)$ is quasiperiodic in R^3:

$$u(x_1 + \omega_1, x_2, x_3) = u(x_1, x_2, x_3) + \omega_1, \; u(x_1, x_2 + \omega_2, x_3) = u(x_1, x_2, x_3),$$
$$u(x_1, x_2, x_3 + \omega_3) = u(x_1, x_2, x_3). \tag{11}$$

The last equalities denote that the external field is fixed in the x_1- direction. The problem (10) is equivalent to the following problem

$$u = (1-\rho)v, \quad \frac{\partial u}{\partial n} = (1+\rho)\frac{\partial v}{\partial n} \quad \text{on } |t| = r, \tag{12}$$

where $\rho := (\lambda_1 - 1)/(\lambda_1 + 1), \quad v(x) := (\lambda_1 + 1)u_1(x)/2.$ The effective conductivity in the x_1- direction is determined by the formula

$$\lambda_e^1 = \lambda_1 \iiint_{D_1} u_1 dx + \iiint_{D} u dx = 1 + 2\rho c \frac{\partial v}{\partial x_1}(0), \tag{13}$$

where $c = \frac{4}{3}\pi r^3$ is the volume fraction of inclusions. Here the Stokes formula and the mean value theorem of the harmonic function theory are applied.

Berdichevskij [14] reduced the problem (10), (11) to an infinite system of linear algebraic equations. This system has been truncated and approximate formulae has been deduced to calculate the effective conductivity. A method of perturbations with respect to the parameter ρ has been proposed in [15]. In the present paper we consider the limit case $\lambda_1 \to \infty \Leftrightarrow \rho \to 1$. In this case the problem (12) becomes the Dirichlet problem $u = 0$ on $|t| = r$, where $u(x)$ is harmonic in D and quasiperiodic, i.e. (11) holds. Introduce the function $w(x) := u(x) - x_1$ harmonic in D It satisfies the Dirichlet problem

$$w(t) = -t_1 \quad \text{on } |t| = r, \tag{14}$$

and periodic . If we know $w(x)$, then $v(x)$ is easily constructed by the Neumann problem $\frac{\partial v}{\partial n} = \frac{1}{2}\frac{\partial u}{\partial n}$ on $|t| = r$ for the ball $|x| < r$. Calculating $\frac{\partial v}{\partial x_1}(0)$ and substituting into (13) we obtain λ_e^1. So the crucial point is concluded in the problem (14).

We apply the formula (5) for (14), when $f(x) = -x_1|x|^{-3}$. If $n \to \infty$ in (5) then we have the formal series

$$w(x) = \sum_{j=0}^{\infty} S_j(x), \tag{15}$$

where $S_0(x) = -\sum_{k=0}^{\infty} f\left(x_k^*\right) = \sum_{k=0}^{\infty}\left(x_1 - a_k^1\right)\left|x - a_k\right|^{-3} := \mathbf{P}_{11}(x) - \boldsymbol{\gamma}_1$, $\mathbf{P}_{11}(x)$ is the Berdichevskij - Weierstrass function, $\boldsymbol{\gamma}_1$ is a Berdichevskij's tensor [14]. The function $\mathbf{P}_{11}(x)$ is periodic and $\mathbf{P}_{11}(x) \sim x_1|x|^{-3}$, as $x \to 0$.

The series (15) corresponds to the generalized method of Schwarz for the lattice **Q**. We can prove convergence of (15) only for small *r*.

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