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Invariant Solutions of the Multidimensional Boussinesq Equation

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Abstract

The reduction of the n-dimensional Boussinesq equation with respect to all subalgebras of rank n of the invariance algebra of this equation is carried out. Some exact solutions of this equation are obtained.

1 Introduction

In this paper, we make research of the Boussinesq equation

$$\frac{\partial u}{\partial x_0} + \nabla\left[(au+b)\nabla u\right] + cu + d = 0, \tag{1}$$

where

$$u = u\left(x_0, x_1, \ldots, x_n\right), \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right),$$

a, b, c, d are real numbers, $a \neq 0$. This equation has applications in hydrology [1, 2] and heat conduction [3]. Group properties of (1) were discussed in [4] for $n = 1$, in [5] for $n = 2, 3$, and in [6, 7] for each n. In the case $n \leq 3$, $c = d = 0$, some invariant solutions of (1) have been obtained in [1, 6–9]. The aim of the present paper is to perform the symmetry reduction of (1) for each n to ordinary differential equations. Using this reduction we find invariant solutions of this equation.

2 Classification of subalgebras of the invariance algebra

The substitution $(au+b) = v^{\frac{1}{2}}$ reduces equation (1) to

$$\frac{\partial v}{\partial x_0} + v^{\frac{1}{2}}\Delta v + \delta v + \gamma v^{\frac{1}{2}} = 0, \tag{2}$$

where $\delta = 2c$, $\gamma = 2ad - bc$. If $\gamma = \delta = 0$, then equation (2) is invariant under the direct sum of the extended Euclidean algebras $A\widetilde{E}\,(1) = \langle P_0, D_1\rangle$ and $A\widetilde{E}\,(n) = \langle P_1, \ldots, P_n\rangle \Supset \left(AO\,(n) \oplus \langle D_2\rangle\right)$, $AO\,(n) = \langle J_{ab}:\ a, b = 1, \ldots, n\rangle$ generated by the vector fields [7]:

$$\begin{aligned} &P_0 = \frac{\partial}{\partial x_0}, \quad D_1 = x_0\frac{\partial}{\partial x_0} - 2v\frac{\partial}{\partial v}, \quad P_b = \frac{\partial}{\partial x_b}, \\ &J_{bc} = x_b\frac{\partial}{\partial x_c} - x_c\frac{\partial}{\partial x_b}, \quad D_2 = x_b\frac{\partial}{\partial x_b} + 4v\frac{\partial}{\partial v}, \end{aligned} \tag{3}$$

where $b, c = 1, \ldots, n$.

If $\gamma = 0$, $\delta \neq 0$, then equation (2) is invariant under $A\widetilde{E}\,(1) \oplus A\widetilde{E}\,(n)$, where [5]

$$P_0 = \mathrm{e}^{\delta x_0}\frac{\partial}{\partial x_0} - 2\delta\,\mathrm{e}^{\delta x_0}\,v\frac{\partial}{\partial v}, \quad D_1 = -\frac{1}{\delta}\frac{\partial}{\partial x_0},$$

whereas the remaining operators are of the form (3). For $\gamma \neq 0$ equation (2) is invariant under the direct sum of $AE\,(1) = \langle P_0 \rangle$ and $AE\,(n) = \langle P_1, \ldots, P_n \rangle \Supset AO\,(n)$ generated by the vector fields (3). From here we assume that $\gamma = 0$.

Let $v = v\,(x_0, x_1, \ldots, x_n)$ be a solution of equation (2) invariant under P_0. In this case if $\delta = 0$, then $v = v\,(x_1, \ldots, x_n)$ is a solution of the Laplace equation $\Delta v = 0$. If $\delta \neq 0$, then

$$v = \mathrm{e}^{-2\delta x_0}\,\varphi\,(x_1, \ldots, x_n)\,, \tag{4}$$

where $\Delta\varphi = 0$. Furthermore for each solution of the Laplace equation, function (4) satisfies equation (2). In this connection, let us restrict ourselves to those subalgebras of the algebra $F = A\widetilde{E}\,(1) \oplus A\widetilde{E}\,(n)$ that do not contain P_0. Among subalgebras possessing the same invariants, there exists a subalgebra containing all the other subalgebras. We call it by the I-maximal subalgebra. To carry out the symmetry reduction of equation (2), it is sufficient to classify I-maximal subalgebras of the algebra F up to conjugacy under the group of inner automorphisms of the algebra F.

Denote by $AO\,[r,s]$, $r \leq s$, a subalgebra of the algebra $AO\,(n)$. It is generated by operators J_{ab}, where $a, b = r, r+1, \ldots, s$. If $r > s$, then we suppose that $AO\,[r,s] = 0$. Let $AE\,[r,s] = \langle P_r, \ldots, P_s \rangle \Supset AO\,[r,s]$ for $r \leq s$ and $AE\,[r,s] = 0$ for $r > s$.

Let us restrict ourselves to those subalgebras of the algebra F whose projections onto $AO\,(n)$ be subdirect sums of algebras of the form $AO\,[r,s]$.

Theorem 1 *Up to conjugacy under the group of inner automorphisms, the algebra F has 7 types of I-maximal subalgebras of rank n which do not contain P_0 and satisfy the above condition for projections:*

$$L_0 = AE(n);$$

$$L_1 = (AO\,(m) \oplus AO\,[m+1,q] \oplus AE\,[q+1,n]) \Supset \langle D_1, D_2 \rangle\,,$$
$$\textit{where} \quad 1 \leq m \leq n-1,\ m+1 \leq q \leq n;$$

$$L_2 = (AO\,(m) \oplus AE\,[m+1,n]) \Supset \langle D_1 + \alpha D_2 \rangle \quad (\alpha \in \mathbb{R},\ 1 \leq m \leq n)\,;$$

$$L_3 = AO\,(m-1) \oplus \{(\langle P_0 + P_m \rangle \oplus AE\,[m+1,n]) \Supset \langle D_1 + D_2 \rangle\} \ \ (2 \leq m \leq n)\,;$$

$$L_4 = \langle P_0 + P_1 \rangle \oplus AE\,[2,n]\,;$$

$$L_5 = AO\,(m) \oplus (AE\,[m+1,n] \Supset \langle D_2 + \alpha P_0 \rangle) \quad (\alpha = 0, \pm 1;\ 1 \leq m \leq n)\,;$$

$$L_6 = \langle J_{12} + P_0,\ D_2 + \alpha P_0 \rangle \uplus AE\,[3,n] \quad (\alpha \in \mathbb{R})\,.$$

Proof. Let K be an I-maximal rank n subalgebra of the algebra F, $\pi(F)$ be a projection of K onto $AO(n)$ and $W = \langle P_0, P_1, \ldots, P_n \rangle \cap K$. If a projection of W onto $\langle P_0 \rangle$ is nonzero, then W is conjugate to $\langle P_0 + P_m, P_{m+1}, \ldots, P_n \rangle$. In this case $\pi(K) = AO(m-1) \oplus AO[m+1,n]$ and a projection of K onto $\langle D_1, D_2 \rangle$ is zero or it coincides with $\langle D_1 + D_2 \rangle$. Therefore $K = L_3$ or $K = L_4$.

Let a projection of W onto $\langle P_0\rangle$ be zero. If $\dim W = n-q$, then up to conjugacy $W = \langle P_{q+1},\dots,P_n\rangle$ and $AE[q+1,n]\subset K$. In this case $\pi(K) = Q\oplus AO[q+1,n]$, where Q is a subalgebra of the algebra $AO(q)$. For $q=0$ we have the algebra L_0. Let $Q\neq 0$. Then Q is the subdirect sum of the algebras $AO[1,m_1]$, $AO[m_1+1,m_2],\dots,AO[m_{s-1}+1,m_s]$, where $m_s\le q$ and in this case a projection of K onto $\langle P_1,\dots,P_n\rangle$ is contained in $\langle P_{m_s+1},\dots,P_n\rangle$. Since the rank of Q doesn't exceed m_s-s, we have $s\le 2$. If $s=2$, then $K=L_1$.

Let $s=1$. If a projection of K onto $\langle P_0\rangle$ is zero, then K is conjugate to L_2 or L_5, where $\alpha=0$. If the projection of K onto $\langle P_0\rangle$ is nonzero, then $K=L_6$ or $K=L_5$, where $\alpha=\pm 1$. The theorem is proved.

3 Reduction of the Boussinesq equation without source

For each of the subalgebras L_1–L_6 obtained in Theorem 1, we point out the corresponding ansatz $\omega' = \varphi(\omega)$ solved for v, the invariant ω, as well as the reduced equation which is obtained by means of this ansatz. In those cases where the reduced equation can be solved, we point out the corresponding invariant solutions of the Boussinesq equation:

3.1. $v = \left(\dfrac{x_1^2+\cdots+x_m^2}{x_0}\right)^2\varphi(\omega)$, $\quad \omega = \dfrac{x_1^2+\cdots+x_m^2}{x_{m+1}^2+\cdots+x_q^2}$, then

$$2\omega^2(1+\omega)\;\ddot\varphi + \left[(8+m)\,\omega-(q-m-4)\,\omega^2\right]\dot\varphi + 2\,(2+m)\,\varphi - \varphi^{\frac12} = 0.$$

The reduced equation has the solution $\varphi = \dfrac{1}{4\,(2+m)^2}$. The corresponding invariant solution of equation (1) is of the form

$$u = \frac{x_1^2+\cdots+x_m^2}{2\,(2+m)\,ax_o} - \frac{b}{a}. \qquad (5)$$

3.2. $v = x_0^{4\alpha-2}\varphi(\omega)$, $\quad \omega = \dfrac{x_1^2+\cdots+x_m^2}{x_0^{2\alpha}}$, then

$$2\omega\ddot\varphi + \left(m-\alpha\omega\varphi^{-\frac12}\right)\dot\varphi + (2\alpha-1)\,\varphi^{\frac12} = 0. \qquad (6)$$

For $\alpha = \dfrac14$, the reduced equation is equivalent to the equation

$$4\omega\dot\varphi + (2m-4)\,\varphi - \omega\varphi^{\frac12} = \widetilde{C},$$

where $\tilde{C}$ is an arbitrary constant. If $\tilde{C}$ =0, then

$$\varphi = \left[\frac{\omega}{2\,(m+2)} + C\omega^{\frac{2-m}{4}}\right]^2.$$

The corresponding invariant solution of the Boussinesq equation is of the form

$$u = \frac{x_1^2+\cdots+x_m^2}{2\,(m+2)\,ax_0} + C\left(x_1^2+\cdots+x_m^2\right)^{\frac{2-m}{4}} x_0^{\frac{m-6}{8}} - \frac{b}{a}.$$

The nonzero function $\varphi = (A\omega + B)^2$, where A and B are constants, satisfies equation (6) if and only if one of the following conditions holds:

1. $\alpha = \frac{1}{2}, \quad A = 0;$
2. $A = \frac{1}{4+2m}, \quad B = 0;$
3. $\alpha = \frac{1}{m+2}, \quad A = \frac{1}{4+2m}.$

By means of φ obtained above, we find the invariant solution (5) and solution

$$u = \frac{x_1^2 + \cdots + x_m^2}{2\,(2+m)\,ax_0} + Bx_0^{-\frac{m}{m+2}} - \frac{b}{a}. \tag{7}$$

3.3. $v = (x_0 - x_m)^2 \varphi(\omega), \quad \omega = \frac{x_1^2 + \cdots + x_{m-1}^2}{(x_0 - x_m)^2}$, then

$$2\omega(\omega+1)\;\ddot{\varphi} + \left(m - 1 - \omega - \omega\varphi^{-\frac{1}{2}}\right)\dot{\varphi} + \varphi + \varphi^{\frac{1}{2}} = 0.$$

3.4. $v = \varphi(\omega)$, $\omega = x_0 - x_1$, then $\varphi^{\frac{1}{2}}\ddot{\varphi} + \dot{\varphi} = 0$. The general solution of the reduced equation is of the form

$$\varphi^{\frac{1}{2}} + \frac{C}{2}\ln\left|C - 2\varphi^{\frac{1}{2}}\right| = -\omega + C'.$$

The corresponding invariant solution of equation (1) is the function $u = u(x_0, x_1)$ given implicitly by

$$au + b + \frac{C}{2}\ln|C - 2au - 2b| = x_1 - x_0 + C',$$

where C and C' are arbitrary constants.

3.5. $v = \left(x_1^2 + \cdots + x_m^2\right)^2 \varphi(\omega), \quad \omega = 2x_0 - \alpha\ln\left(x_1^2 + \cdots + x_m^2\right)$, then

$$2\alpha^2\ddot{\varphi} + \left[\varphi^{-\frac{1}{2}} + (m+6)\,\alpha\right]\dot{\varphi} + 2\,(m+2)\,\varphi = 0.$$

3.6. $v = \left(x_1^2 + x_2^2\right)^2 \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2\right)^{-\frac{\alpha}{2}} \exp\left(x_0 + \arctan\frac{x_1}{x_2}\right)$, then

$$\left(1 + \alpha^2\right)\omega^2\ddot{\varphi} + \left[\omega\varphi^{-\frac{1}{2}} + \left(\alpha^2 - 8\alpha + 1\right)\omega\right]\dot{\varphi} + 16\varphi = 0.$$

Notation. Case 3.1 corresponds to the subalgebra L_1; 3.2 – to L_2 and so on.

4 Reduction of the Boussinesq equation with source

Case 4.1 corresponds to the subalgebra L_1; 4.2 – to L_2 and so on. Let $2ad - bc = 0$.

4.1. $v = \left(x_1^2 + \cdots + x_m^2\right)^2 \varphi(\omega), \quad \omega = \left(x_1^2 + \cdots + x_m^2\right)\left(x_{m+1}^2 + \cdots + x_q^2\right)^{-1}$, then

$$2\omega^2(1+\omega)\ddot{\varphi} + \left[(8+m)\omega - (q-m-4)\omega^2\right]\dot{\varphi} + 2(2+m)\varphi + \delta\varphi^{\frac{1}{2}} = 0.$$

4.2. $v = \exp(-4\alpha\delta x_0)\varphi(\omega), \quad \omega = \left(x_1^2 + \cdots + x_m^2\right)\exp(-4\alpha\delta x_0)$, then

$$2\omega\ddot{\varphi} + \left[m + \alpha\delta\omega\varphi^{-\frac{1}{2}}\right]\dot{\varphi} + \delta(1-2\alpha)\varphi^{\frac{1}{2}} = 0. \tag{8}$$

Integrating this reduced equation for $\alpha = \dfrac{1}{4}$, we obtain the following equation:

$$2\omega\dot{\varphi} + (m-2)\varphi + \frac{1}{2}\delta\omega\varphi^{\frac{1}{2}} = C.$$

For $C = 0$ we have

$$\varphi^{\frac{1}{2}} = \tilde{C}\omega^{\frac{2-m}{4}} - \frac{\delta}{2(m+2)}\omega.$$

The corresponding invariant solution of the Boussinesq equation is of the form

$$u = -\frac{c}{(m+2)a}\left(x_1^2 + \cdots + x_m^2\right) + \tilde{C}\left(x_1^2 + \cdots + x_m^2\right)^{\frac{2-m}{4}}\exp\left(-\frac{2+m}{4}cx_0\right) - \frac{b}{a},$$

where $\tilde{C}$ is an arbitrary constant.

If $\alpha = \dfrac{1}{2+m}$, then $\varphi^{\frac{1}{2}} = -\dfrac{\delta}{2(m+2)}\omega + B$ is a solution of equation (7). Thus, we find the exact solution

$$u = B'\exp\left(-\frac{4}{2+m}cx_0\right) - \frac{c}{(m+2)a}\left(x_1^2 + \cdots + x_m^2\right) - \frac{b}{a}.$$

4.3. $v = \left(x_m\,\mathrm{e}^{-\delta x_0} + \dfrac{1}{\delta}\,\mathrm{e}^{-2\delta x_0}\right)^2\varphi(\omega), \quad \omega = \dfrac{x_1^2 + \cdots + x_{m-1}^2}{\left(x_m + \frac{1}{\delta}\,\mathrm{e}^{-\delta x_0}\right)^2}$, then

$$2\omega(1+\omega)\ddot{\varphi} + (m-1-6\omega)\dot{\varphi} = 0.$$

Integrating this reduced equation we obtain

$$\varphi = C_1\int \omega^{\frac{1-m}{2}}(1+\omega)^{\frac{m+5}{2}}\,d\omega + C_2.$$

For $m = 3$ we have the invariant solution of equation (1):

$$u = \frac{1}{a}\left(x_3 + \frac{1}{2c}\,\mathrm{e}^{-2cx_0}\right)\mathrm{e}^{-2cx_0}\left\{C_1\left[\ln\omega + 4\omega + 3\omega^2 + \frac{4}{3}\omega^3 + \frac{\omega^4}{4}\right] + C_2\right\} - \frac{b}{a},$$

with $C_1 \neq 0$, C_2 being arbitrary constants and $\omega = \dfrac{x_1^2 + x_2^2}{\left(x_3 + \frac{1}{2c}\,\mathrm{e}^{-2cx_0}\right)^2}$.

4.4. $v = \mathrm{e}^{-2\delta x_0}\,\varphi(\omega)\,,\quad \omega = x_1 + \dfrac{1}{\delta}\,\mathrm{e}^{-\delta x_0}$, then $\ddot{\varphi} - \varphi^2\dot{\varphi} = 0$. This equation is equivalent to the equation

$$\dot{\varphi} - \frac{\varphi^3}{3} = C'.$$

If $C' = 0$, then $\varphi = \dfrac{\sqrt{3}}{(C - 2\omega)^{\frac{1}{2}}}$. Thus, we find the exact solution

$$u = \frac{3^{\frac{1}{6}}}{a}\frac{\mathrm{e}^{-2cx_0}}{\left(\tilde{C} - 2x_1 - \frac{1}{c}\,\mathrm{e}^{-2cx_0}\right)^{\frac{1}{4}}} - \frac{b}{a}.$$

4.5. $v = \dfrac{\left(x_1^2 + \cdots + x_m^2\right)^2}{\mathrm{e}^{2\delta x_0}}\varphi(\omega)\,,\quad \omega = \alpha\delta\ln\left(x_1^2 + \cdots + x_m^2\right) + 2\,\mathrm{e}^{-\delta x_0}$, then

$$4\alpha^2\delta^2\ddot{\varphi} + 2\alpha\delta\,(m+6)\,\dot{\varphi} - 2\delta\varphi^{\frac{1}{2}}\dot{\varphi} + 4\,(m+2)\,\varphi = 0.$$

For $\alpha = 0$ we find the exact solution of equation (1):

$$u = \left(x_1^2 + \cdots + x_m^2\right)\mathrm{e}^{-2cx_0}\left[\tilde{C} + \frac{m+2}{ac}\,\mathrm{e}^{-2cx_0}\right] - \frac{b}{a}.$$

4.6. $v = \left(x_1^2 + x_2^2\right)^2\mathrm{e}^{-2\delta x_0}\,\varphi(\omega)\,,\quad \omega = \left(x_1^2 + x_2^2\right)^{\frac{\alpha}{2}}\exp\left(\dfrac{1}{\delta}\,\mathrm{e}^{-\delta x_0} - \arctan\dfrac{x_1}{x_2}\right)$, then

$$\left(1 + \alpha^2\right)\omega^2\ddot{\varphi} + \left[\left(\alpha^2 + 8\alpha + 1\right)\omega - \omega\varphi^{-\frac{1}{2}}\right]\dot{\varphi} + 16\varphi = 0.$$

5 Application to heat conduction of non-linear materials

The exact solution of the Boussinesq equation obtained in the previous section can be applied to calculate the temperature distribution in metals.

Heat conduction of platinum is described by the coefficient of heat conduction [3]

$$\lambda_{PT}(u) = 0,0156u + 68,75$$

depending on the temperature u. Function (7) written for $m = 3$ and $B = -1$ as

$$u(t,r) = 64,103r^2t^{-1} - t^{-\frac{3}{5}} - 4,407\cdot 10^3$$

describes the temperature distribution in a platinum ball

$$r^2 \le 1 \qquad \left(x_1^2 + x_2^2 + x_3^2 \le 1\right),$$

when the external boundary temperature is of the form

$$u(t,1) = 64,103t^{-1} - t^{-\frac{3}{5}} - 4,407\cdot 10^3.$$

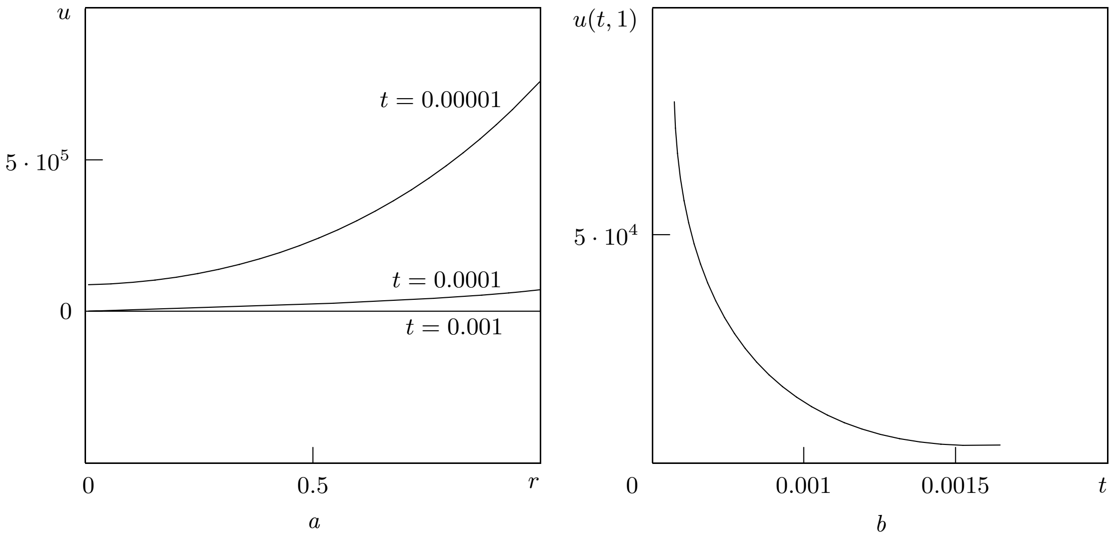

Fig.1. Temperature distribution $u(t,r)$ of a platinum ball: *a)* temperatures $u(t,r)$ at the times t; *b)* boundary temperature $u(t,r)$ at the time $[0; 0,002]$.

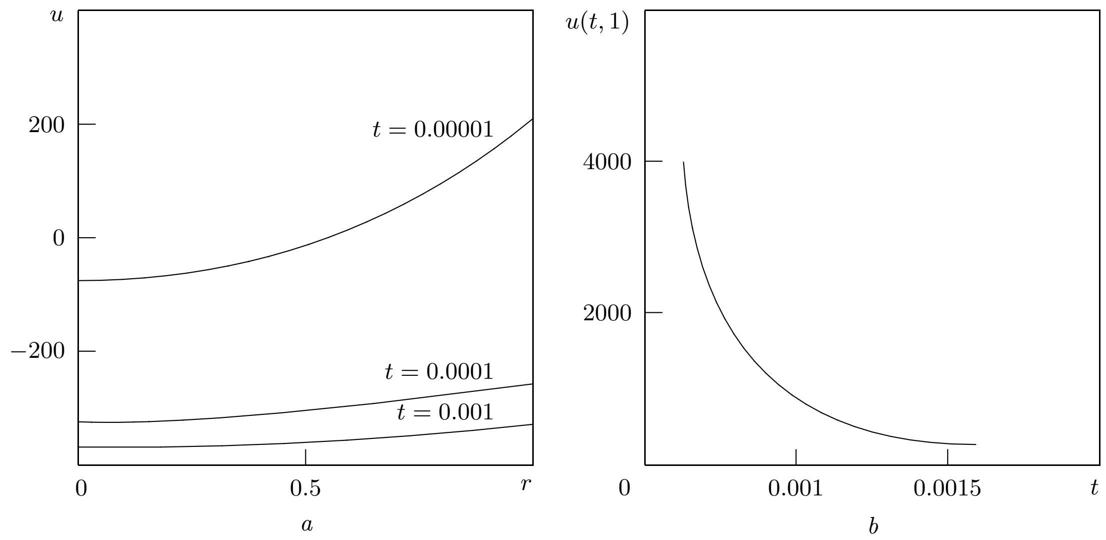

Fig.2. Temperature distribution $u(t,r)$ of a beryllium ball: *a)* temperatures $u(t,r)$ at the times t; *b)* boundary temperature $u(t,r)$ at the time $[0; 0,002]$.

Heat conduction of beryllium is described by the heat conduction coefficient [3]

$$\lambda_B(u) = \frac{1}{3}u + 158$$

depending on the temperature u. Function (7) written for $m = 3$ and $B = -2$ as

$$u(t,r) = \frac{3r^2}{10t} - 2t^{-\frac{3}{5}} - 474$$

describes the temperature distribution in a beryllium ball

$$r^2 \le 1 \qquad \left(r^2 = x_1^2 + x_2^2 + x_3^2\right),$$

when the external boundary temperature is of the form

$$u\,(t,1) = \frac{3}{10}t^{-1} - 2t^{-\frac{3}{5}} - 474.$$

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