# Effective conductivity of square array of cylinders. Exact formula

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**Abstract**

The $\mathbb{R}$–linear conjugation problem for a disk in a class of doubly periodic functions, i.e., $\mathbb{R}$–linear problem on torus, has been solved in the form of series by Eisenstein's functions. The result is applied to calculation of the effective conductivity of composites with circular inclusions. The results of the present paper are also presented in [12], [11], [19].

# 1 R–linear problem on torus

Let $\mathbb{Z}[i]$ be the set of complex numbers with integer real and imaginary parts. Consider a square lattice $\mathcal{Q}$ which is defined by two fundamental translation vectors expressed by complex numbers 1 and $i$ on the complex plane $\mathbf{C}$. Introduce the zero-th cell $Q_0 := \{z = t_1 + it_2 : -1/2 < t_j < 1/2 \ (j = 1, 2)\}$. The lattice $\mathcal{Q}$ consists of the cell $Q_m := \{z \in \mathbf{C} : z - m \in Q_0\}$, where $m \in \mathbb{Z}[i]$. Let the disk $D_1 = \{z \in \mathbf{C} : |z| < r\}$ lies in the cell $Q_0$, $D$ be the complement of the closure of $D_1$ to $Q_0$. To find a function $\psi(z)$ analytic in $D, D_1$, continuous in the closure of the considered domains with the following conjugation condition

$$\psi^-(t) = \psi^+(t) + \rho \left(\frac{r}{t}\right)^2 \overline{\psi^+(t)}, \quad |t| = r \tag{1.1}$$

where $\psi(z)$ is doubly periodic with respect to $\mathcal{Q}$

$$\psi(z+1) = \psi(z) = \psi(z+i). \tag{1.2}$$

A given constant $\rho$ satisfies the inequality $-1 < \rho < 1$. The problem (1.1)–(1.2) can be considered as the homogeneous $\mathbb{R}$–linear problem for the unit circle on the torus represented by the cell $Q_0$. It can be also considered as an $\mathbb{R}$–linear problem for infinitely connected domain bounded by $|t - m| = r$ ($m = m_1 + im_2 \in \mathbb{Z}[i]$, i.e., $m_1$ and $m_2$ are integers).

In mechanics, the problem (1.1)–(1.2) corresponds to a problem for a composite material, when the conductivity of the matrix is normalized by unity and $\lambda_1 = \frac{1+\rho}{1-\rho}$ is the respective conductivity of the inclusions.

# 2 Classical Eisenstein-Rayleigh sums and Eisenstein series for the square lattice

In the present section we introduce the fundamental constants and functions of the elliptic function theory following Weil [23] and Akhiezer [2].

The Eisenstein summation method is defined as follows

$$\sum_{m_1,m_2} = \lim_{N\to\infty} \sum_{m_2=-N}^{m_2=N} \left( \lim_{M\to\infty} \sum_{m_1=-M}^{m_1=M} \right). \tag{2.1}$$

Using this summation we introduce the conditionally convergent sum

$$S_2 := \sum_{m_1,m_2} {}'(m_1 + im_2)^{-2} = \sum_m {}'m^{-2}, \tag{2.2}$$

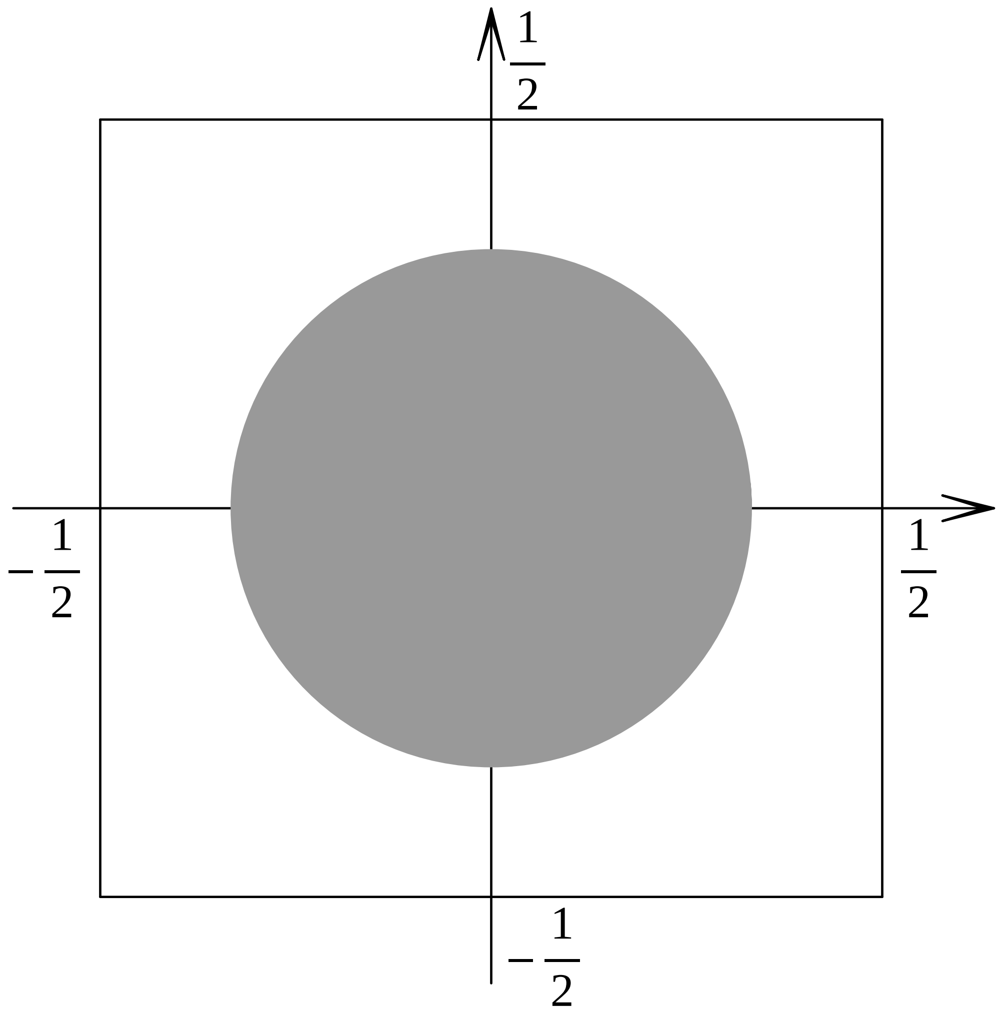


Figure 1: Square cell

where $m_1$ and $m_2$ run over all integer numbers except the pair $m_1 = m_2 = 0$. It is known [13] that $S_2 = \pi$. Following Eisenstein and Rayleigh we introduce the absolutely convergent sums

$$S_n := \sum_m {}' m^{-n}, \ n = 3, 4, \dots . \tag{2.3}$$

For the square array, it is known that $S_{2n+1} = 0$ and $S_{4n+2} = 0$ for $n \in \mathbb{N}$.

The Eisenstein series are defined as follows

$$E_n(z) := \sum_m (z - m)^{-n}, \ n = 2, 3, \dots . \tag{2.4}$$

The Eisenstein summation method (2.1) is applied to $E_2(z)$. The series $E_n(z)$ for $n = 3, 4, \dots$ as a function in $z$ converge absolutely and almost uniformly in the domain $\mathbf{C}\backslash\mathbb{Z}[i]$. Each of the functions (2.4) is doubly periodic and has a pole of order $n$ at $z = 0$.

The Eisenstein series and the Weierstrass function $\mathcal{P}(z)$ are related by the identities

$$E_2(z) = \mathcal{P}(z) + \pi, \tag{2.5}$$

$$E_n(z) = \frac{(-1)^n}{(n-1)!} \frac{d^{n-2}\mathcal{P}(z)}{dz^{n-2}}, \quad n = 3, 4, \dots \tag{2.6}$$

The Eisenstein functions of the even order $E_{2n}(z)$ can be presented in the form of the series

$$E_{2n}(z) = \frac{1}{z^{2n}} + \sum_{k=0}^{\infty} \sigma_k^{(n)} z^{2(k-1)}, \tag{2.7}$$

where

$$\sigma_k^{(n)} = \frac{(2n + 2k - 3)!}{(2n - 1)!(2k - 2)!} S_{2(n+k-1)}. \tag{2.8}$$

Computationally efficient formulas to calculate (2.3) has been proposed in [12] for rectangular array of cylinders. Here we write these formulas for the square array.

$$S_4 = \frac{1}{3}\pi^4 \left( \frac{1}{15} + 16 \sum_{m=1}^{\infty} \frac{m^3 h^{2m}}{1 - h^{2m}} \right) \approx 3.151212,$$

where $h = \exp(-\pi)$. The other sums are calculated by the recursion formula

$$S_{2k} = \frac{3}{(2k+1)(2k-1)(k-3)} \sum_{m=2}^{k-2} (2m-1)(2k-2m-1) S_{2m} S_{2(k-m)}.$$

Let us compute the first sums in terms of $S_4$

$$S_8 = \frac{3}{7} S_4^2 \approx 4.255773, \; S_{12} = \frac{18}{143} S_4^3 \approx 3.938849,$$

$$S_{16} = \frac{9}{221} S_4^4 \approx 4.015695, \; S_{20} = \frac{54}{4199} S_4^5 \approx 3.996097.$$

# 3 Reduction the R–linear problem to a functional equation

In the present section we reduce the $\mathbb{R}$–linear problem (1.1)–(1.2) to a functional equation. First, we introduce the operator

$$T_m \psi(z) := \left( \frac{r}{z-m} \right)^2 \left( \overline{\psi \left( \frac{r^2}{\overline{z-m}} \right)} - \overline{\psi(0)} \right), \tag{3.1}$$

where $m \in \mathbb{Z}[i]$.

Introduce the Banach space $C_A$ of functions continuous in $|z| \leq r$ and analytic in $|z| < r$ with the norm $||\psi(z)|| = \max_{|t|=r} |\psi(z)|$.

**Theorem 3.1 ([17]).** *Let $\sum_m$ and $\sum_m'$ denote Eisenstein's summation, respectively with the term $m = 0$ and without it.*

*(i) The series*

$$\Psi_0(z) = \sum{}_m' T_m \psi(z)$$

*converges absolutely and uniformly in the closure of the cell $Q_0$ for each $\psi \in C_A$ to a function analytic in $Q_0$ and continuous in its closure.*

*(ii) The series*

$$\Psi(z) = \sum_m T_m \psi(z) \tag{3.2}$$

*converges absolutely and uniformly in each compact subset of $D$ to a function analytic in $D$ continuous in its closure and doubly periodic with respect to the considered lattice.*

*(iii) The linear operator*

$$T = \sum{}_m' T_m \tag{3.3}$$

*is compact in $C_A$.*

Using Eisenstein's summation one can rewrite (3.2) in the form

$$\Psi(z) = \sum_m \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} - \overline{\psi(0)} r^2 E_2(z), \tag{3.4}$$

since the sum

$$\Psi_e(z) = \sum_m \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} \tag{3.5}$$

is correctly defined by Eisenstein's summation. It is commutative with integrals, it can be differentiated terms by terms as the absolutely and uniformly convergent sum (3.2). However, it is forbidden to change the order of summation in (3.5).

We present the unknown function $\psi(z)$ in the form of its Taylor expansion

$$\psi(z) = \sum_{k=0}^{\infty} \psi_k z^k. \tag{3.6}$$

Then

$$\overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} = \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} \frac{1}{(z-m)^k} \tag{3.7}$$

for each $m$. Substitution of (3.7) in (3.5) yields

$$\Psi_e(z) = \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} E_{k+2}(z). \tag{3.8}$$

Introduce the function

$$\Phi(z) = \begin{cases} \psi(z) - \rho r^2 \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} (E_{k+2}(z) - z^{-k-2}), \ |z| \le r, \\ \\ \psi(z) - \rho r^2 \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} E_{k+2}(z), \ z \in D, \end{cases} \tag{3.9}$$

analytic in $D$ and in the disk $|z| < r$. Calculate the jump of $\Phi(z)$ across $|t| = r$

$$\Delta = \Phi^+(t) - \Phi^-(t) = \psi^+(t) - \psi^-(t) + \rho \left(\frac{r}{t}\right)^2 \overline{\psi^-(t)}. \tag{3.10}$$

Using (1.1) one can see that $\Delta = 0$. Applying the principle of analytic continuation and Liouville's theorem we obtain that $\Phi(z)$ is a constant, say $c$. This complex constant corresponds to the vector of the external flux applied to composite [14], [15]. Then the definition of $\Phi(z)$ in $|z| \le r$ yields the following functional equation with respect to $\psi \in C_A$

$$\psi(z) = \rho r^2 \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} (E_{k+2}(z) - z^{-k-2}) + c, \ |z| \le r, \tag{3.11}$$

which can be also written in the form

$$\psi(z) = \rho \sum{}'_m \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\overline{\frac{r^2}{z-m}}\right)} + c, \ |z| \le r. \tag{3.12}$$

We have

$$\overline{\psi\left(\overline{\frac{r^2}{z-m}}\right)} = \overline{\psi(z^* - m)},$$

where $z^* = \frac{r^2}{\overline{(z-m)}} + m$ is the inversion with respect to the circle $|z-m| = r$. The inversion $z^*$ transforms the disk $|z| \le r$ onto a closed disk $D^*$ from $|z-m| < r$ when $m \ne 0$. The translation $z \mapsto z - m$ returns $D^*$ to $|z| < r$. Therefore, the shift $\frac{r^2}{\overline{z-m}}$ maps $|z| \le r$ onto a closed disk lying in $|z| < r$. Equations with shifts into domain are called iterative functional equations [6].

**Theorem 3.2 ([17]).** *Equation (3.11) with $|\rho| < 1$ has a unique solution in $C_A$. This solution can be found by the method of successive approximations converging in $C_A$, i.e., uniformly convergent in $|z| \le r$.*

# 4 Solution to functional equation

We note in the previous section that it is possible to apply the method of successive approximations to the functional equation (3.11) and hence to construct an approximate solution in symbolic form as it was done in [4, 14, 21]. However, in this case of one inclusion in the periodicity cell it is possible also to write explicitly each term of this approximation, i.e., it is possible to write exact solution of the functional equation in the form of the series with explicitly written terms. It is convenient to perform it using the functional equation (3.12).

We are looking for $\psi(z)$ in the form of the series

$$\psi(z) = c \sum_{k=0}^{\infty} \left(\rho r^2\right)^k \psi_k(z). \tag{4.1}$$

Then (3.12) yields the following recurrence relations

$$\psi_0(z) = 1,$$

$$\psi_k(z) = \sum{}'_m \frac{1}{(z-m)^2} \overline{\psi_{k-1}\left(\overline{\frac{r^2}{z-m}}\right)}, \ k = 1, 2, \dots \ . \tag{4.2}$$

Applying (4.2) we obtain $\psi_k(z)$ in the form of conditionally convergent series [16]

$$\psi_k(z)=\sum{}'_{\nu_1}\sum{}'_{\nu_2}\cdots\sum{}'_{\nu_k}(\nu_1-z)^{-2}\left(\overline{\nu_2}-\frac{r^2}{\nu_1-z}\right)^{-2}\left(\nu_3-\frac{r^2}{\overline{\nu_2}-\frac{r^2}{\nu_1-z}}\right)^{-2}$$
$$\dots\left(\overline{\nu_k}-\frac{r^2}{\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\dots}}}\right)^{-2}, \tag{4.3}$$

where $\nu_l$ corresponds to $m$ from (4.2). For definiteness (4.3) is written for even $k$. Using the Eisenstein series we simplify (4.3).

We apply (2.4), (2.7) and (2.8) to the latest term from (4.3)

$$\sum{}'_{\nu_k}\left(\overline{\nu_k}-\frac{r^2}{\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\dots}}}\right)^{-2}=\sum_{n_1=1}^{\infty}\sigma_{n_1}^{(1)}\frac{r^{4(n_1-1)}}{\left(\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\dots}}\right)^{2(n_1-1)}}. \tag{4.4}$$

Substitution of (4.4) in (4.3) yields

$$\psi_k(z)=\sum_{n_1=1}^{\infty}\sigma_{n_1}^{(1)}r^{4(n_1-1)}\sum{}'_{\nu_1,\nu_2\dots\nu_k}(\nu_1-z)^{-2}\left(\overline{\nu_2}-\frac{r^2}{\nu_1-z}\right)^{-2}\left(\nu_3-\frac{r^2}{\overline{\nu_2}-\frac{r^2}{\nu_1-z}}\right)^{-2}$$
$$\dots\left(\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\dots}}\right)^{-2n_1}. \tag{4.5}$$

We now apply (2.4), (2.7) and (2.8) to the latest term from (4.5)

$$\sum{}'_{\nu_{k-1}}\left(\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\nu_{k-3}-\frac{r^2}{\dots}}}\right)^{-2n_1}=\sum_{n_2=1}^{\infty}\sigma_{n_2}^{(n_1)}\frac{r^{4(n_2-1)}}{\left(\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\dots}}\right)^{2(n_2-1)}}. \tag{4.6}$$

Then (4.5) becomes

$$\psi_k(z)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sigma_{n_1}^{(1)}\sigma_{n_2}^{(n_1)}r^{4(n_1+n_2-2)}\dots$$
$$\sum{}'_{\nu_1,\nu_2\dots\nu_{k-2}}(\nu_1-z)^{-2}\left(\overline{\nu_2}-\frac{r^2}{\nu_1-z}\right)^{-2}\dots\left(\overline{\nu_{k-2}}-\frac{r^2}{\nu_{k-3}-\frac{r^2}{\dots}}\right)^{-2n_2}. \tag{4.7}$$

We again apply (2.4), (2.7) and (2.8) to (4.7) and so forth. At the end we obtain the desired formula

$$\psi(z)=c\left(1+\sum_{k=1}^{\infty}(\rho r^2)^k\sum_{n_1,n_2,\dots n_k}\sigma_{n_1}^{(1)}\sigma_{n_2}^{(n_1)}\dots\sigma_{n_{k-1}}^{(n_{k-2})}E_{n_k}(z)r^{4(n_1+n_2+\dots+n_k-k)}\right). \tag{4.8}$$

Here we use (4.1).

The effective conductivity of the square array is determined by the following equality [14] (there is also a proof that $\psi(0)/c$ is real)

$$\widehat{\lambda} = 1 + 2\rho\pi r^2 \frac{\psi(0)}{c}. \tag{4.9}$$

Substitution of (4.1) and (4.8) to (4.9) yields the exact formula

$$\widehat{\lambda} = 1+2\rho\pi r^2+2\pi \sum_{k=1}^{\infty} \rho^{k+1} \sum_{n_1,n_2,\ldots n_k} \sigma_{n_1}^{(1)}\sigma_{n_2}^{(n_1)} \ldots \sigma_{n_{k-1}}^{(n_{k-2})}\sigma_1^{(n_k)} r^{4(n_1+n_2+\cdots+n_k)-2(k-1)}, \tag{4.10}$$

where $\sigma_n^{(k)}$ is given by (2.8), $n_j$ run over unit to infinity in the sum.

# 5 Conclusion

The formula (4.10) can be considered as an expansion of $\widehat{\lambda}$ on the concentration of the inclusions $\pi r^2$ and the contrast parameter $\rho$. This analiticity is consistent with the previous general result of Bergman [3]. The formula (4.10) includes all known formulas for $\widehat{\lambda}$ approximated by $\pi r^2$ and $\rho$. However, in the case when $r \to 1/2$ and $\rho \to 1$ the series (4.10) diverges to $+\infty$. Asymptotic formulas for $\widehat{\lambda}$ in this case were obtained by McPhedran et al. [7] (see also papers cited therein). Direct application of (4.10) to this limit case is doubtful. One can find some notes on application of the functional equations to this case in [18].

It could be interesting to estimate $\psi(0)$ without direct solution to the problem (1.1) or to the functional equation (3.11) in the limit case. The general theory of bounds for $\widehat{\lambda}$ is presented by Milton [8] without an address to the $\mathbb{R}$–linear problem.

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