

# Representative Cell in Mechanics of Composites and Generalized Eisenstein-Rayleigh Sums

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## Abstract

Consider two-dimensional two-component periodic composite made from a collection of non-overlapping, identical, circular disks, embedded in a matrix. The effective conductivity tensor can be written in the form of expansion on "basic elements" which depend only on locations of the disks. These elements are expressed in terms of the Eisenstein series. The representative cell of a composite is defined as the minimal size periodicity cell corresponding to the set of basic elements calculated for the composite. An algorithm to determine the representative cell for a given composite is constructed.

*Keywords:* representative element; representative volume element; Eisenstein series; boundary value problem; effective conductivity

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## 1 Introduction

One of the most important notation of composite materials is the *representative volume element* (RVE). One can give a vague physical definition of

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this term as follows. RVE is a part of material which is small enough from a macroscopical point of view and can be thus treated as a typical element of the heterogeneous medium. On the other hand, it is sufficiently large in the microscopical scale, and it represents typical microstructure of the material under consideration. In the present paper we first give a rigorous definition of the representative element. Then, we determine its minimal size. The geometrical interpretation of the problem is shown in Figures 1 and 2. The large cell  $Q'_{(0,0)}$  presented in Figure 1 is replaced by a smaller one,  $Q_{(0,0)}$  (see Figure 2) with three inclusions per the periodicity cell.

Drugan and Willis [6] estimated the minimal RVE size of the composite with uniformly distributed spherical inclusions considering the ensemble average of stress and strain "at finite" and "at infinite length". The infinite length corresponds to the overall elastic constants of the homogenized materials. RVE was introduced as a finite element the mean constants of which are numerically closed to the mean constants at the infinite length. The authors derived quantitative estimates for the minimal RVE size in the case when matrix is reinforced by a random dispersion of non-overlapping identical spheres. Gusev [7] numerically calculated the overall elastic constants of individual Monte Carlo realizations with 8, 27 and 64 spheres and established that the scatter among the individual elastic constants is very small for the same uniform distribution of spheres. Adler [1] discussed questions of the reconstruction of porous media by statistical data and numerically constructed RVE. Kolodziej [10] systematically applied the collocation method to compute effective conductivity of various two-dimensional composites.

In the present paper we consider two-dimensional two-component periodic composite medium made from a collection of non-overlapping, identical, circular disks, embedded in an otherwise uniform matrix. We discuss fields governed by the Laplace equation, when the inclusions have scalar conductivity  $\lambda$  and separated by a matrix of unit conductivity. Let  $\rho = (\lambda - 1)/(\lambda + 1)$  be the contrast parameter introduced by Bergman [4]. Mityushev [13], Berlyand and Mityushev [5] established that the effective conductivity tensor  $\Lambda_e$  of the considered composites has the form of double series on the concentration of inclusions and on "basic elements" which depend only on locations of the inclusions. These basic elements are written in terms of the Eisenstein series [16]. Coefficients in the double series depend on  $\rho$ . We say that two composites are equivalent if expansions of their  $\Lambda_e$  have the same basic elements. Therefore, we divide the set of the composites with circular identical

inclusions onto classes of equivalence determined only by geometrical structure of the composite. In particular, composites with the same locations of inclusions but with different  $\rho$  belong to the same class of equivalence. Note that composites belonging to a class of equivalence can have different  $\Lambda_e$ ; and composites from different classes can have the same  $\Lambda_e$ . Each composite material is represented by a periodicity cell. In each class of equivalence we choose a composite having the minimal size cell. This cell is called the representative cell of the considered class of equivalent composite materials.

We propose a constructive algorithm to determine the representative cell for any distribution of inclusions using only pure geometrical parameters. More precisely, at the beginning we calculate the *generalized Eisenstein-Rayleigh sums* depending on the centers of circular inclusions for given large cell. Then using these sums we construct the (minimal) representative cell, i.e., we calculate its fundamental translation vectors and determine the positions of inclusions within this cell.

We use the elliptic (doubly periodic meromorphic) functions in the form of the *Eisenstein series* introduced by Eisenstein in 1847 and developed by Weil [16]. The classical lattice sums (the Eisenstein sums) were applied to calculation of the effective conductivity tensor by Rayleigh [11] when a representative cell contains one inclusion. In Section 2 we recall the classical Eisenstein-Rayleigh sums, the Eisenstein series and introduce the generalized Eisenstein-Rayleigh sums. The latter sums can be considered as a generalization of the classical sums to a cell with few inclusions. Section 3 is devoted to the effective conductivity tensor represented by a cell with few identical circular disks. In Section 4 we describe a method how to replace a periodic composite by another one with the same effective properties but with smaller size of the periodicity cell. Numerical examples illustrating the general theory are presented in Subsections 4.2 and 4.3.

## 2 Generalized Eisenstein-Rayleigh sums

### 2.1 Classical Eisenstein-Rayleigh sums

In the present subsection we introduce the fundamental parameters of the elliptic function theory following Weil [16] and Akhiezer [2]. Consider a lattice  $\mathcal{Q}$  which is defined by two fundamental translation vectors expressed by complex numbers  $\omega_1$  and  $\omega_2$  on the complex plane  $\mathbb{C}$ . For the definiteness

we assume that  $Im\tau > 0$ , where  $\tau = \omega_2/\omega_1$ . Introduce the  $(0,0)$  - cell  $Q_{(0,0)} := \{z = t_1\omega_1 + t_2\omega_2 : -1/2 < t_j < 1/2 (j = 1,2)\}$ . The lattice  $\mathcal{Q}$  consists of the cell  $Q_{(m_1,m_2)} := \{z \in \mathbb{C} : z - m_1\omega_1 - m_2\omega_2 \in Q_{(0,0)}\}$ , where  $m_1$  and  $m_2$  run over integer numbers.

The Eisenstein summation method is defined as follows

$$\sum_{m_1, m_2} = \lim_{N \rightarrow \infty} \sum_{m_2 = -N}^{m_2 = N} \left( \lim_{M \rightarrow \infty} \sum_{m_1 = -M}^{m_1 = M} \right). \quad (1)$$

Using this summation we introduce the conditionally convergent sum

$$S_2(\omega_1, \omega_2) := \sum_{m_1, m_2} '(m_1\omega_1 + m_2\omega_2)^{-2}, \quad (2)$$

where  $m_1$  and  $m_2$  run over all integer numbers except the pair  $m_1 = m_2 = 0$ . The sum (2) is slowly convergent. An efficient in computations formula for  $S_2(\omega_1, \omega_2)$  was deduced in [12]

$$S_2(\omega_1, \omega_2) = \frac{2}{\omega_1} \zeta\left(\frac{\omega_1}{2}\right), \quad (3)$$

where  $\zeta(z)$  is the Weierstrass  $\zeta$  - function. Rylko [15] deduced another efficient formula

$$S_2(\omega_1, \omega_2) = \left(\frac{\pi}{\omega_1}\right)^2 \left(\frac{1}{3} - 8 \sum_{m=1}^{\infty} \frac{mh^{2m}}{1-h^{2m}}\right), \text{ where } h = \exp(\pi i\tau). \quad (4)$$

Following Eisenstein and Rayleigh we introduce the absolutely convergent sums

$$S_n(\omega_1, \omega_2) := \sum_{m_1, m_2} '(m_1\omega_1 + m_2\omega_2)^{-n}, \quad n = 3, 4, \dots \quad (5)$$

It is known that  $S_n(\omega_1, \omega_2) = 0$  for odd  $n$ . For even  $n$  the Eisenstein - Rayleigh sums (5) can be easily calculated through the rapidly convergent infinite sums

$$g_2(\omega_1, \omega_2) : = \left(\frac{\pi}{\omega_1}\right)^4 \left(\frac{4}{3} + 320 \sum_{m=1}^{\infty} \frac{m^3 h^{2m}}{1-h^{2m}}\right), \quad (6)$$

$$g_3(\omega_1, \omega_2) : = \left(\frac{\pi}{\omega_1}\right)^6 \left(\frac{8}{27} - \frac{448}{3} \sum_{m=1}^{\infty} \frac{m^5 h^{2m}}{1-h^{2m}}\right). \quad (7)$$

Then

$$S_4(\omega_1, \omega_2) = \frac{g_2(\omega_1, \omega_2)}{60}, \quad S_6(\omega_1, \omega_2) = \frac{g_3(\omega_1, \omega_2)}{1400}. \quad (8)$$

The sums  $S_{2n}(\omega_1, \omega_2)$  ( $n \geq 4$ ) are calculated by the recurrence formula

$$S_{2n}(\omega_1, \omega_2) = \frac{3}{(2n+1)(2n-1)(n-3)} \sum_{m=2}^{n-2} (2m-1)(2n-2m-1) S_{2m} S_{2(n-m)}. \quad (9)$$

## 2.2 Eisenstein series

In the present subsection we summarize the main facts of the Eisenstein series theory following Weil [16]. The Eisenstein series are defined as follows

$$E_n(z; \omega_1, \omega_2) := \sum_{m_1, m_2} (z - m_1\omega_1 - m_2\omega_2)^{-n}, \quad n = 2, 3, \dots \quad (10)$$

The Eisenstein summation method (1) is applied to  $E_2(z; \omega_1, \omega_2)$ . The series  $E_n(z; \omega_1, \omega_2)$  for  $n = 3, 4, \dots$  as a function in  $z$  converge absolutely and almost uniformly in the domain  $\mathbb{C} \setminus \cup_{m_1, m_2} (m_1\omega_1 + m_2\omega_2)$ . Each of the functions (10) is doubly periodic and has a pole of order  $n$  at  $z = 0$ . However, further it will be convenient to define the value of  $E_n(z; \omega_1, \omega_2)$  at the point zero as follows

$$E_n(0; \omega_1, \omega_2) := S_n(\omega_1, \omega_2). \quad (11)$$

The Eisenstein series and the Weierstrass function  $\mathcal{P}(z; \omega_1, \omega_2)$  are related by the identities

$$E_2(z; \omega_1, \omega_2) = \mathcal{P}(z; \omega_1, \omega_2) + S_2(\omega_1, \omega_2), \quad (12)$$

$$E_n(z; \omega_1, \omega_2) = \frac{(-1)^n}{(n-1)!} \frac{d^{n-2}}{dz^{n-2}} \mathcal{P}(z; \omega_1, \omega_2). \quad (13)$$

## 2.3 Generalized Eisenstein-Rayleigh sums

We now proceed to introduce one of the most important mathematical object of the present paper, the generalized Eisenstein-Rayleigh sums. Consider a set of points  $a_k$  ( $k = 1, 2, \dots, N$ ) in the cell  $Q_{(0,0)}$ . Let  $p$  be a natural

number;  $k_s$  runs over 1 to  $N$ ,  $n_j = 2, 3, \dots$ . Let  $\mathbf{C}$  be the operator of complex conjugation. The value

$$e_{n_1 \dots n_p}(\omega_1, \omega_2) := \frac{1}{N^{p+1}} \sum_{k_0 k_1 \dots k_p} E_{n_1}(a_{k_0} - a_{k_1}) \overline{E_{n_2}(a_{k_1} - a_{k_2})} \dots \mathbf{C}^p E_{n_p}(a_{k_{p-1}} - a_{k_p}) \quad (14)$$

is called the generalized Eisenstein-Rayleigh sum. The parameters  $\omega_1$  and  $\omega_2$  are omitted in  $E_n$ .

For instance, for  $p = 1$  (14) implies

$$e_n(\omega_1, \omega_2) := \frac{1}{N^2} \sum_{k_0, k_1} E_n(a_{k_0} - a_{k_1}). \quad (15)$$

According to (11)  $e_n(\omega_1, \omega_2)$  becomes the classical Eisenstein-Rayleigh sum  $S_n(\omega_1, \omega_2)$  in the case  $N = 1$ .

All sums and series introduced in this section are constructed basing on the fixed fundamental translation vectors  $\omega_1$  and  $\omega_2$ . We are also interested in the normalized Eisenstein series

$$E_n(z; 1, \tau) := \sum_{m_1, m_2} (z - m_1 - m_2 \tau)^{-n}, \quad n = 2, 3, \dots \quad (16)$$

We have the relations

$$E_n(z; \omega_1, \omega_2) = \omega_1^{-n} E_n\left(\frac{z}{\omega_1}; 1, \tau\right), \quad (17)$$

$$e_{n_1 \dots n_p}(\omega_1, \omega_2) = \omega_1^{-2k} e_{n_1 \dots n_p}(1, \tau), \quad (18)$$

where  $2k := n_1 + \dots + n_p$ . Note that we shall need further only the even sums  $n_1 + \dots + n_p$ .

### 3 Structure of the effective conductivity tensor

The results of the present section are based on the papers [13], [5]. Consider the cell  $Q_{(0,0)}$  with  $N$  non-overlapping circular disks  $D_k$  of the radius  $r$  with the centers  $a_k \in Q_{(0,0)}$  ( $k = 1, 2, \dots, N$ ). Let  $D_0$  be the complement of the closure of all disks  $D_k$  to  $Q_{(0,0)}$ . We study the conductivity of the doubly periodic composite material, when the domains  $D_{per} :=$

$\cup_{(m_1, m_2)}(D_0 \cup \partial Q_{(0,0)} + m_1\omega_1 + m_2\omega_2)$  and  $D_k + m_1\omega_1 + m_2\omega_2$  ( $m_1, m_2$  are integers) are occupied by materials of conductivities  $\lambda_0$  and  $\lambda$ , respectively. The conductivity of the inclusions  $\lambda$  is expressed relative to  $\lambda_0$ . Hence, the conductivity of the matrix can be takes as unity ( $\lambda_0 = 1$ ). The local potential  $u(z)$  in  $Q_{(0,0)}$  satisfies the conjugation conditions

$$u^+(t) = u^-(t), \quad \frac{\partial u^+}{\partial n}(t) = \lambda \frac{\partial u^-}{\partial n}(t) \quad \text{on } \partial D_k = \{t \in \mathbb{C} : |t - a_k| = r\}, \quad k = 1, 2, \dots, N, \quad (19)$$

where  $\frac{\partial}{\partial n}$  is the outward normal derivative and, for instance,

$$u^+(t) := \lim_{\substack{z \rightarrow t, \\ z \in D_0}} u(z), \quad u^-(t) := \lim_{\substack{z \rightarrow t, \\ z \in D_k}} u(z). \quad (20)$$

The potential  $u(z)$  satisfies the quasi-periodicity conditions

$$u(z + \omega_1) = u(z) + \Omega_1, \quad u(z + \omega_2) = u(z) + \Omega_2. \quad (21)$$

Here, the function  $u(z)$  is harmonic in  $Q_{(0,0)}$  except  $\partial D_k$  ( $k = 1, 2, \dots, N$ ), the circles  $\partial D_k$  are orientated in the clockwise direction. Equations (19) model the perfect contact between matrix and inclusions. Equations (21) mean that the external field has the gradient  $(\Omega_1, \Omega_2)$  in the coordinates based on the vectors  $\omega_1$  and  $\omega_2$ . In order to determine the effective conductivity tensor  $\Lambda_e$  it is sufficiently to solve problem (19), (21) with two linear independent vectors  $(\Omega_1, \Omega_2)$ .

The effective conductivity tensor  $\Lambda_e$  of the considered composite has the following structure

$$\Lambda_e = (1 + 2\rho v)\mathbf{I} + 2\rho v \sum_{k=1}^{\infty} \mathbf{P}_k v^k, \quad (22)$$

where  $v = \frac{N\pi r^2}{|Q_{(0,0)}|}$  is the concentration of the disks in the cell  $Q_{(0,0)}$ ,  $|Q_{(0,0)}|$  is the area of  $Q_{(0,0)}$ ,  $\mathbf{I}$  is the identity tensor,

$$\mathbf{P}_k = \begin{pmatrix} \operatorname{Re} A_k & \operatorname{Im} A_k \\ \operatorname{Im} A_k & C_k \end{pmatrix}, \quad (23)$$

$$A_k = |Q_{(0,0)}|^k \sum_{n_1 \dots n_p} B_{n_1 \dots n_p}^{(k)} e_{n_1 \dots n_p}(\omega_1, \omega_2),$$

The constants  $B_{n_1 \dots n_p}^{(k)}$  depend only on  $k$ ,  $\rho$  and  $n_1, \dots, n_p$ . Here,  $n_j = 2, 3, \dots$ ;  $k = 1, 2, \dots$ . The values  $C_k$  have an analogous form. Only the terms  $e_{n_1 \dots n_p}(\omega_1, \omega_2)$  depend on the centers of inclusions  $a_k$  in the representation (22)-(23) of  $\Lambda_e$ . Few first coefficients  $A_k$  have the form

$$\begin{aligned}
A_1 &= \frac{\rho}{\pi} e_2, & A_2 &= \frac{\rho^2}{\pi^2} e_{22}, & A_3 &= \frac{1}{\pi^3} [-2\rho^2 e_{33} + \rho^3 e_{222}], \\
A_4 &= \frac{1}{\pi^4} [3\rho^2 e_{44} - 2\rho^3 (e_{332} + e_{233}) + \rho^4 e_{2222}], \\
A_5 &= \frac{1}{\pi^5} [-4\rho^2 e_{55} + \rho^3 (3e_{442} + 6e_{343} + 3e_{244}) - \\
&\quad + 2\rho^4 (e_{3322} + e_{2332} + e_{2233}) + \rho^5 e_{22222}], \\
A_6 &= \frac{1}{\pi^6} [5\rho^2 e_{66} - \rho^3 (4e_{255} + 12e_{354} + 12e_{453} + 4e_{552}) + \rho^4 (3e_{2244} + \\
&\quad + 6e_{2343} + 4e_{3333} + 3e_{2442} + 6e_{3432} + 3e_{4422}) - 2\rho^5 (e_{22233} + e_{22332} + \\
&\quad + e_{23322} + e_{33222}) + \rho^6 e_{222222}],
\end{aligned} \tag{24}$$

where the argument  $(\omega_1, \omega_2)$  is omitted. It follows from (23) that each coefficient  $A_k$  involves a set of  $e_{n_1 \dots n_p}(\omega_1, \omega_2)$ . The mapping  $k \mapsto (n_1, \dots, n_p)$  has been precisely described in [13], [5]. In particular, the following conditions are fulfilled:

- i)  $n_1 + \dots + n_p = 2k$ ;
- ii)  $n_j \leq k - p$  ( $j = 1, 2, \dots, p$ );
- iii)  $n_j \geq 2$  ( $j = 1, 2, \dots, p$ ).

Hence, in order to calculate the tensor  $\Lambda_e$  up to  $O(v^{L+1})$  we have to find  $A_k$  ( $k = 1, 2, \dots, L - 1$ ). Therefore, we have to calculate a finite number of the generalized Eisenstein-Rayleigh sums  $e_{n_1 \dots n_p}(\omega_1, \omega_2)$ .

## 4 Representative cell

### 4.1 General equations

Consider a large fundamental region  $Q'_{(0,0)}$  constructed by the fundamental translation vectors  $\omega'_1$  and  $\omega'_2$ . Let  $Q'_{(0,0)}$  contains  $N'$  non-overlapping circular disks  $D'_k$  of the radius  $r$  with the centers  $a'_k \in Q_{(0,0)}$  ( $k = 1, 2, \dots, N'$ ). Let  $\Lambda'_e$  be the effective conductivity tensor of the composite material represented by the region  $Q'_{(0,0)}$  with inclusions  $D'_k$ . We are interested in the following question. To replace  $Q'_{(0,0)}$  by another small cell  $Q_{(0,0)}$  which contains inclusions



$D_k = \{z \in \mathbb{C} : |z - a_k| < r\}$  ( $k = 1, 2, \dots, N$ ) and which has an effective conductivity tensor  $\Lambda_e$  closed to  $\Lambda'_e$ . We assume that the concentration  $v$  of the inclusions in both materials is the same. Closeness is defined by the accuracy  $O(v^{L+1})$  for the difference  $\Delta\Lambda_e = \Lambda_e - \Lambda'_e$  with prescribed  $L$ . We say that  $Q_{(0,0)}$  is a representative cell for the region  $Q'_{(0,0)}$  with the accuracy  $O(v^{L+1})$  if  $\Delta\Lambda_e = O(v^{L+1})$ . We say that  $Q_{(0,0)}$  is the minimal representative cell for the region  $Q'_{(0,0)}$  if  $Q_{(0,0)}$  is a representative cell with minimal possible area  $|Q_{(0,0)}|$ . For brevity we further call the minimal representative cell by the representative cell. The existence of the representative cell is evident, since in the worst case one can take  $Q_{(0,0)} = Q'_{(0,0)}$ .

We adopt the designations (22), (23) for the representative cell. Consider the corresponding formulas for  $\Lambda'_e$

$$\Lambda'_e = (1 + 2\rho v)\mathbf{I} + 2\rho v \sum_{k=1}^{\infty} \mathbf{P}'_k v^k, \quad (25)$$

$$A'_k = |Q'_{(0,0)}|^k \sum_{n_1 \dots n_p} B_{n_1 \dots n_p}^{(k)} e_{n_1 \dots n_p}(\omega'_1, \omega'_2). \quad (26)$$

Note that the coefficients  $B_{n_1 \dots n_p}^{(k)}$  has the same form in (23) and (26).  $\Delta\Lambda_e$  is of order  $O(v^{L+1})$  if  $A'_k = A_k$  for  $k = 1, 2, \dots, L - 1$ . Therefore,  $\Delta\Lambda_e$  is of order  $O(v^{L+1})$  if and only if

$$|Q_{(0,0)}|^k e_{n_1 \dots n_p}(\omega_1, \omega_2) = |Q'_{(0,0)}|^k e_{n_1 \dots n_p}(\omega'_1, \omega'_2) \quad (27)$$

for  $k = 1, 2, \dots, L - 1$  and corresponding sets of the numbers  $n_1, \dots, n_p$ . According to our definition  $Q_{(0,0)}$  is a representative cell for the region  $Q'_{(0,0)}$  with the accuracy  $O(v^{L+1})$  if and only if the relations (27) are fulfilled.

One can consider (27) as a system of equations with respect to  $\omega_1, \omega_2, a_1, a_2, \dots, a_N$  including the unknown number  $N$  with the restriction  $|a_j - a_m| \leq 2r$  ( $j \neq m$ ). One can assume that one of the centers, say  $a_N$ , lies at the origin, since geometrically any cell is determined up to translation. The fundamental region  $Q_{(0,0)}$  as well as the translation vectors  $\omega_1, \omega_2$  can be chosen by infinitely many ways [2]. For any doubly periodic structure on the plane it is always possible to construct such a pair  $\omega_1, \omega_2$  that  $\omega_1 > 0$  and  $Im\tau > 0$ .

The area of  $Q_{(0,0)}$  is calculated by  $\omega_1$  and  $\omega_2$

$$|Q_{(0,0)}| = \omega_1^2 Im\tau. \quad (28)$$

On the other hand we also have

$$|Q_{(0,0)}| = \frac{N\pi r^2}{v} \quad (29)$$

that yields the formula

$$\omega_1 = \sqrt{\frac{N\pi r^2}{vIm\tau}}. \quad (30)$$

In order to construct the representative cell with the prescribed accuracy  $O(v^{L+1})$  we propose to solve the system (27) with fixed  $L$  increasing the number of inclusions in the cell  $N$  from 1 to  $N'$ . Then  $N$  is fixed in each step of the study of (27).

Applying (18) and (28) we rewrite (27) in the form

$$(Im\tau)^k e_{n_1\dots n_p}(1, \tau) = |Q'_{(0,0)}|^k e_{n_1\dots n_p}(\omega'_1, \omega'_2), \quad k = 1, 2, \dots, L-1. \quad (31)$$

We can consider (31) as a system with respect to  $\tau, a_1, a_2, \dots, a_{N-1}$  ( $a_N = 0$ ) with the restriction  $|a_j - a_m| \geq 2r$  ( $j \neq m$ ). The right-hand part of (31) is known. If we know a solution of (31), we can calculate  $\omega_1$  from (30).

It is also possible to state the problem of the representative cell with prescribed form of the cell  $Q_{(0,0)}$ . Let us consider the case when  $Q_{(0,0)}$  is a rectangle. Then  $\tau = i\alpha$ , where  $\alpha$  is positive and (30) implies

$$\omega_1 = \sqrt{\frac{N\pi r^2}{\alpha v}}. \quad (32)$$

Equations (31) become

$$\alpha^k e_{n_1\dots n_p}(1, i\alpha) = |Q'_{(0,0)}|^k e_{n_1\dots n_p}(\omega'_1, \omega'_2), \quad k = 1, 2, \dots, L-1. \quad (33)$$

It is hard to investigate analytically the systems (31), (33) in general form, since it is difficult to extract independent equations from the sets (31), (33). In the next subsection simple examples of (33) are considered.

## 4.2 Two inclusions in the representative cell

Consider two inclusion in a rectangular cell ( $N = 2$ ). In this case the positions of the inclusions are determined by one complex parameter  $a = a_2 - a_1$ . By direct calculation one can check that

$$e_n(1, \tau) = \begin{cases} \frac{1}{2}(S_n(1, \tau) + E_n(a; 1, \tau)) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad (34)$$

$$e_{mn}(1, \tau) = e_m(1, \tau) \overline{e_n(1, \tau)}, \quad e_{mnp}(1, \tau) = e_m(1, \tau) \overline{e_n(1, \tau)} e_p(1, \tau), \quad (35)$$

and so on. Therefore, instead of the general Eisenstein-Rayleigh sums in (33) it is sufficient to consider equations with simple sums

$$\alpha^k e_k(1, i\alpha) = |Q'_{(0,0)}|^k e_k(\omega'_1, \omega'_2), \quad k = 2, 4, \dots \quad (36)$$

Substitution of (34) into (36) in the case  $\tau = i\alpha$  yields

$$\alpha^k [S_k(1, i\alpha) + E_k(a; 1, i\alpha)] = p_k, \quad k = 2, 4, \dots, \quad (37)$$

where

$$p_k = 2 |Q'_{(0,0)}|^k e_k(\omega'_1, \omega'_2)$$

are known constants. (37) with  $k = 2, 4$  becomes

$$\alpha^2 [S_2(1, i\alpha) + E_2(a; 1, i\alpha)] = p_2, \quad \alpha^4 [S_4(1, i\alpha) + E_4(a; 1, i\alpha)] = p_4. \quad (38)$$

Using the relations [16], [2]

$$\begin{aligned} E_4(z; 1, i\alpha) &= \frac{1}{6} \frac{d^2}{dz^2} E_2(z; 1, i\alpha), \\ \frac{d^2}{dz^2} E_2(z; 1, i\alpha) &= 6 (E_2(a; 1, i\alpha) - S_2(1, i\alpha))^2 - \frac{1}{2} g_2(1, i\alpha) \end{aligned}$$

and (8) we obtain

$$E_4(z; 1, i\alpha) = (E_2(a; 1, i\alpha) - S_2(1, i\alpha))^2 - 5S_4(1, i\alpha). \quad (39)$$

Then the second equation (38) is transformed to the following one

$$\alpha^4 [(E_2(a; 1, i\alpha) - S_2(1, i\alpha))^2 - 4S_4(1, i\alpha)] = p_4. \quad (40)$$

We express  $E_2(a; 1, i\alpha)$  from the first equation (38)

$$E_2(a; 1, i\alpha) = \frac{p_2}{\alpha^2} - S_2(1, i\alpha) \quad (41)$$

and substitute it in (40)

$$\alpha^4 \left[ \left( \frac{p_2}{\alpha^2} - 2S_2(1, i\alpha) \right)^2 - 4(S_4(1, i\alpha)) \right] = p_4. \quad (42)$$

The latter equation is real, since  $S_2$  and  $S_4$  are real for rectangular arrays. Hence, we have obtained the real number equation (42) with respect to real unknown  $\alpha$ .

Let us consider a numerical example with  $p_2 = 10$ ,  $p_4 = 50$ ,  $r = 0.15$ ,  $v = 0.3$ . Equation (42) has the solution  $\alpha = 0.820$ . Substituting  $\alpha$  in (41) and solving the obtained equation with respect to  $a$  we get  $a = 0.331$ . Then (32) implies  $\omega_1 = 0.758$ . Therefore, the representative cell is described by the fundamental vectors  $\omega_1 = 0.934$ ,  $\omega_2 = i0.820$  with two inclusions with the centers  $a_1 = 0$  and  $a_2 = 0.331$ .

### 4.3 Three inclusions in the representative cell

Consider now three inclusion in the cell ( $N = 3$ ). In this case the positions of the inclusions are determined by two complex parameter  $a_1$  and  $a_2$  ( $a_3 = 0$ ). It follows from equation (30) that

$$\omega_1^2 = c_0(Im\tau)^{-1}, \quad (43)$$

where  $c_0 = 3\pi r^2/v$ . Consider the following equations (31)

$$(Im\tau)^k e_{2k}(1, \tau) = p_{2k}, \quad k = 1, 2, 3, \quad (44)$$

where  $e_{2k}$  are calculated by (15). For numerical computations it is convenient to use formulas (12) involving the Weierstrass function  $\mathcal{P}(z)$  and its derivatives. Then  $e_{2k}(1, \tau)$  become

$$e_2(1, \tau) = S_2(1, \tau) + \frac{2}{9} (\mathcal{P}(a_1) + \mathcal{P}(a_2) + \mathcal{P}(a_1 - a_2)), \quad (45)$$

$$e_4(1, \tau) = -3S_4(1, \tau) + \frac{2}{9} (\mathcal{P}^2(a_1) + \mathcal{P}^2(a_2) + \mathcal{P}^2(a_1 - a_2)), \quad (46)$$

$$e_6(1, \tau) = \frac{1121}{3} S_6(1, \tau) + 6S_4(1, \tau) (\mathcal{P}(a_1) + \mathcal{P}(a_2) + \mathcal{P}(a_1 - a_2)) - \frac{22}{45} (\mathcal{P}^3(a_1) + \mathcal{P}^3(a_2) + \mathcal{P}^3(a_1 - a_2)). \quad (47)$$

Here we use the relations (12) and the following formulas from [2], [16]

$$E_4(z) = \frac{1}{6} \mathcal{P}''(z) = \mathcal{P}^2(z) - 5S_4(1, \tau), \quad (48)$$

$$E_6(z) = -\frac{11}{5}\mathcal{P}^3(z) + 27S_4(1, \tau)\mathcal{P}(z) + 560S_6(1, \tau). \quad (49)$$

Therefore, we arrive at three equations (44) where  $e_{2k}$  have the form (45)–(47) with respect to three unknowns  $a_1$ ,  $a_2$  and  $\tau$ .

Consider a numerical example in which the large cell  $Q'_{(0,0)}$  (see Figure 1) with  $N' = 60$  inclusions of the radius  $r = 0.12$  and the concentration  $v = 0.1$  is determined by the translation vector  $\omega'_1 = 4$ ,  $\omega'_2 = 4i$ . In this case  $p_2 = 0.78 - 0.66i$ ,  $p_4 = -2.15 + 2.27i$ ,  $p_6 = -6.28 - 51i$ . The cell  $Q'_{(0,0)}$  is replaced by a smaller cell  $Q_{(0,0)}$  with  $N = 3$  inclusions. In order to find parameters of  $Q_{(0,0)}$  we solve the system (44). One of the solutions has the form  $a_1 = -0.92$ ,  $a_2 = -0.36 + 0.36i$ ,  $\tau = 0.06 + 0.39i$ . Then (30) yields  $\omega_1 = 1.08$ . The cell  $Q_{(0,0)}$  is presented in Figure 2.

## 5 Conclusion and discussion

In the present paper a rigorous theory of the representative cell in mechanics of periodic composites is proposed. We restrict ourselves by conductivity of two-dimensional two-component composite materials made from a collection of non-overlapping, identical, circular disks, arbitrary embedded in a matrix. The definition of the representative cell is based on the representation of the effective conductivity tensor (22)–(23). We say that the cells  $Q_{(0,0)}$  and  $Q'_{(0,0)}$  are equivalent if they have the same basic elements in the representation (22)–(23). Thus all composites are divided onto classes of equivalences. The minimal size cell in each class is called the representative cell. The basic elements of (22)–(23) are expressed in terms of the generalized Eisenstein-Rayleigh sums (14).

The problem of determination of the representative cell is reduced to the finite system of equation (31) with respect to parameters of the representative cell. Partial cases of this system are considered.

We investigate here the problem of numerical solution to the system (31) with  $N = 2$  and  $N = 3$ . We can give some remarks about the general system (31). First, it is evident that it has infinite number of solution, since any doubly periodic structure is determined by infinite number of the pairs of the fundamental translation vectors. Moreover, as it follows from relations (34)–(35), the system (31) can contain redundant equations. Note also, that for rectangular cell a symmetric map of the location of inclusions could not change the effective conductivity tensor. Hence, if the fundamental

translation vectors are fixed the system (31) with respect to  $a_1, a_2, \dots, a_{N-1}$  can have non-unique solution.

In the present paper we discuss the conductivity two-dimensional problem with circular inclusions. Now we briefly explain why the method can be developed and applied to other problems of the theory of composites.

First, let us consider the same problem but with inclusions having another shape  $S$ . Any plane domain  $S$  can be approximated by packing disks of the radius  $r$ . This approximation can be expressed by appropriate conditions on the centers of the packing disks  $b_1, b_2, \dots, b_P$ . We write them in the form of the constrains on  $b_j$

$$b_j - b_1 = B_j e^{i\psi}, \quad j = 3, 4, \dots, P, \quad |B_2| = |b_2 - b_1|. \quad (50)$$

Here, the constants  $B_2, B_3, \dots, B_P$  are given,  $\psi = \arg \frac{B_2}{b_2 - b_1}$ . The constrains (50) mean that the points  $b_1, b_2, \dots, b_P$  are tied and may only translate and rotate as a stiff body. We can replace all inclusions (say  $M$  inclusions per the cell) by a set of points  $a_1, a_2, \dots, a_N$  divided onto  $M$  subsets each of them contains  $P$  points. We assume that points of each subset satisfy the constrains (50), i. e., each subset of the disks approximates an inclusion of the form  $S$ . These constrains on  $a_1, a_2, \dots, a_N$  should be added to equations (31) in order to obtain a system of equations corresponding to the representative cell with  $M$  inclusions of the shape  $S$ .

The proposed method could be also applied to elastic problems. In addition to the classical Eisenstein functions we should consider the following series [8], [14]

$$E_{pq}(z) = \sum_{m_1, m_2} = \sum_{m_1, m_2} (z - m_1 - im_2)^{-p} \overline{(z - m_1 - im_2)^{-q}}. \quad (51)$$

The lattice sums (51) were presented in [14] in rapidly convergent forms by Fourier transform methods.

Berdichevskij [3] constructed three-dimensional counterparts of the elliptic functions which could be used for three-dimensional conductivity and elasticity problems. Huang [9] proposed exact integral formulas for three-dimensional lattice sums. His examples show that simple quadrature rules with modest numbers of nodes yield highly accurate results. One can find a review of the various numerical methods to calculate three-dimensional lattice sums in [9].

## References

- [1] P. M. Adler (1992) *Porous media. Geometry and transport*, Butterworth/ Heinemann
- [2] N. I. Akhiezer (1990), *Elements of the Theory of Elliptic Functions*, AMS, Rhode Island.
- [3] V. L. Berdichevskij (1983), *Variational principles of continuum mechanics*, Moscow, Nauka [in Russian].
- [4] D. J. Bergman, Bulk physical properties of composite media. In: *Les Méthodes de l'homogénéisation: théorie et applications en physique*, Edition Eyrolles, Paris, (1985) 1-128.
- [5] L. Berlyand, V. Mityushev (2001), Generalized Clausius-Mossotti formula for random composite with circular fibers, *J Statist. Phys.* v.102, N 1/2, 115-145.
- [6] Drugan, W.J., Willis, J.R (1996) A micromechanics-based nonlocal constitutive equation and estimates of representative volume element size for elastic composites, *Journal of the Mechanics and Physics of Solids*, v. 44, 497-524.
- [7] Gusev A. A. 1997, Representative volume element size for elastic composites: a numerical study, *Journal of the Mechanics and Physics of Solids*, v. 45, 1449-1459.
- [8] J. Helsing (1994), Bounds on the shear modulus of suspensions by interface method, *J Mech. Phys. Solid*, v. 42, 1123-1138.
- [9] Huang J. 1999, Integral representations of harmonic lattice sums, *J Math Phys*, v. 40, 5240-5246.
- [10] J. A. Kolodziej (1987), Calculation of the effective thermal conductivity of the unidirectional composites. *Arch. Termodynamiki* v.8, 101-107 [in Polish]
- [11] Lord Rayleigh (1892), On the influence of obstacles arranged in rectangular order upon the properties of medium, *Phil. Mag.*, 34, pp. 481-502.

- [12] V. Mityushev (1997), Transport properties of regular arrays of cylinders, ZAMM 77, pp. 115–120.
- [13] V. Mityushev, Transport properties of doubly periodic array of circular cylinder and optimal design problem, Appl.Math Optim 44 (2001) 17–31
- [14] A. B. Movchan, N. A. Nicorovici, R. C. McPhedran (1997), Green's tensors and lattice sums for elastostatics and elastodynamics, Proc. Roy. Soc. London Ser. A 453, no. 1958, 643–662.
- [15] N. Rylko, Transport properties of the regular array of highly conducting cylinders. J Engrg Math 38 (2000) 1-12.
- [16] A. Weil (1976), Elliptic Functions According to Eisenstein and Kronecker, Springer-Verlag, Berlin etc.



**Figures**

1. Large cell  $Q'_{(0,0)}$  with 60 inclusions.
2. Small cell  $Q_{(0,0)}$  with three inclusions centered at  $a_1 = -0.92$ ,  $a_2 = -0.36 + 0.36i$ ,  $a_3 = 0$ . It represents the large cell from the previous picture.



