Complex Variables
Vol. 50, No. 7–11, 10 June–15 September 2005, 621–630

R-linear problem on torus and its applications to composites†

VLADIMIR MITYUSHEV*

Department of Mathematics, Pedagogical Academy, ul. Podcharazych 2,
Cracow 30-084, Poland

Communicated by O. Çelebi

(*Received 16 February 2005*)

We study the structure of the general solution of the **R**-linear conjugation problem with constant coefficients in a class of doubly periodic functions, i.e., the **R**-linear problem on torus. The results are applied to determine the effective conductivity tensor of composites.

Keywords: R-linear problem; Boundary value problem; Effective conductivity

AMS 2000 Mathematics Subject Classifications: 30E25; 74Q15

1. Introduction

Let D_k be mutually disjoint simply connected domains in the complex plane $\mathbf{C}$ bounded by smooth curves ∂D_k $(k = 1, 2, \ldots, n)$, D be the complement of all closures of ∂D_k to the extended complex plane $\mathbf{C} \cup \{\infty\}$. Let ∂D_k be orientated in counter clockwise direction. Let $a(t)$, $b(t)$ and $c(t)$ be given Hölder continuous functions on $\partial D = -\cup_{k=1}^{n} \partial D_k$; $a(t) \neq 0$.

The **R**-linear conjugation problem on **C** is stated as follows. To find a function $\varphi(z)$ analytic in $D, D_1, \ldots, D_n$, continuous in the closures of the considered domains with the following conjugation condition

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + c(t), \quad t \in \partial D. \tag{1}$$

*Email: mityu@ipgp.jussieu.fr
†Dedicated to H. Begehr on the occasion of his 65th birthday.

Complex Variables
ISSN 0278-1077 print: ISSN 1563-5066 online © 2005 Taylor & Francis Group Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/02781070500086982

In the case $b(t) = 0$ we arrive at the **C-*linear conjugation problem*** [1]

$$\varphi^+(t) = a(t)\varphi^-(t) + c(t), \quad t \in \partial D. \tag{2}$$

Nöther's theory for the problem (1) has been constructed by Mikhajlov [2] by its reduction to a singular integral equation. In the case $a(t) \equiv b(t)$ the problem (1) is reduced to the Riemann–Hilbert problem. One can find the solution of the latter problem for simply connected domains in [1] and for multiply connected domains in [3]. The qualitative theory of the Riemann–Hilbert problem for generalized analytic functions is presented by Wen and Begehr [4]. Dzuraev [5] and Komyak [6,7] investigated a relation between the **R**-linear problem and two-dimensional singular integral equations. Litvinchuk and Spitkovsky [8] studied the **R**-linear problem for a circle by reducing it to a two-dimensional **C**-linear problem.

The **C**-linear problem (2) in a class of doubly periodic functions, i.e., on torus has been solved in [9] by Zverovich's method. In the present article we consider the **R**-linear problem for multiply connected domains on torus with constant coefficients normalized as follows $a(t) = 1$, $b(t) = \rho_k$ on each ∂D_k. The elliptic case $|\rho_k| < 1$ on terminology of [2] is considered. This problem is equivalent to the main plane conductivity problem of the theory of composites, the so-called cell periodicity problem [10–12]. In section 1 we state three problems on torus: the cell periodicity problem, an **R**-linear problem with constant coefficients, a problem which is obtained from the latter one by differentiation along boundary. In section 2 we investigate the relations between these three problems. On the plane **C** the question of equivalence of the problems is simple [3] since solutions of these problems are distinguished by additive constants. On torus the operator of integration transforms periodic functions to functions having constant jumps across the periodicity cell. We describe these additional constants and explain their physical meaning. In section 4 using the above results we deduce a formula to calculate the effective conductivity tensor in terms of the complex potentials. The obtained results justify the formulas of the articles [13,14] devoted to composites with circular inclusions, where the above constants were *a priori* fixed on the base of the physical arguments.

2. Statement of the problems

Consider a square lattice $\mathcal{Q}$ which is defined by two fundamental translation vectors expressed by complex numbers 1 and i on the complex plane **C** of variable $z = x + iy$. Let $m = m_1 + im_2$ denote complex numbers with integer real and imaginary parts m_1 and m_2. Introduce the m–cell $Q_m := \{z = t_1 + it_2 : -1/2 < t_j < 1/2 \ (j = 1, 2)\}$. The lattice $\mathcal{Q}$ consists of the cell $Q_m := \{z \in \mathbf{C} : z - m \in Q_0\}$. Consider mutually disjoint domains $D_k = \{z \in \mathbf{C} : |z - a_k| < r_k\}$ with Lyapunov's boundary ∂D_k $(k = 1, 2, \ldots, n)$ lying in the zero-th cell Q_0. Let D be the complement of the closure of all D_k to Q_0. Let $\partial/\partial n$ denote the outward normal derivative to the curves ∂D_k. Then according to the direction of the normal vector the signs "+" and "−" are assigned to the domains D_k and D, respectively.

2.1. *Conductivity problem*

Let λ_k $(k = 1, 2, \ldots, n)$ be given positive constants, c_1 and c_2 be given real constants. To find a function $u(z)$ harmonic in D and D_k $(k = 1, 2, \ldots, n)$, continuously differentiable in the closure of the considered domains with the following conjugation condition

$$u^-(t) = u^+(t), \tag{3}$$

$$\frac{\partial u^-}{\partial n}(t) = \lambda_k \frac{\partial u^+}{\partial n}(t), \quad t \in \partial D_k, \quad k = 1, 2, \ldots, n \tag{4}$$

and the quasi-periodicity relations with respect to the lattice $\mathcal{Q}$

$$u(z+1) - u(z) = c_1, \quad u(z+i) - u(z) = c_2. \tag{5}$$

The latter problem is a cell periodicity problem in the theory of composites [10–12], when the domains D called by matrix and D_k called by inclusion are occupied materials with the conductivities $\lambda = 1$ and λ_k, respectively.

Theorem 1 [10–12] *Conductivity Problem has a unique solution up to an arbitrary additive real constant*, *say* C.

The relations (3) and (4) have a simple physical interpretation. For instance, in the heat conduction $u(z)$ is the distribution of temperature and $\lambda^{\pm}\partial u^{\pm}/\partial n$ is the normal flux from $D^{\pm}$ on the boundary of inclusion. Therefore, (3) implies that the interior and exterior to the inclusion, boundary values of the temperature coincide; (4) implies that the normal flux on ∂D_k is preserved. Hence, (3)–(4) model the perfect thermal contact between inclusion and matrix. The vector (c_1, c_2) models the external gradient applied to the composite.

Introduce the constant $\rho_k = (\lambda_k - 1)(\lambda_k + 1)^{-1}$ satisfying the inequality $-1 < \rho_k < 1$, since λ_k is positive.

2.2. *Problem R*

To find a function $\varphi(z)$ analytic in D, D_k, continuously differentiable in the closure of the considered domains with the following condition

$$\varphi^-(t) = \varphi^+(t) - \rho_k \overline{\varphi^+(t)}, \quad t \in \partial D_k, \ k = 1, 2, \ldots, n, \tag{6}$$

$$\varphi(z+1) - \varphi(z) = c_1 + id_1, \quad \varphi(z+i) - \varphi(z) = c_2 + id_2, \tag{7}$$

where c_1 and c_2 are given real constants, d_1 and d_2 are undetermined real constants which should be found.

Problem R can be considered as an **R**-linear problem on the torus represented by the cell Q_0. It can also be considered as an **R**-linear problem for infinitely connected domain bounded by $\partial D_k + m$ ($m = m_1 + im_2$, m_1 and m_2 run over integers).

Consider another **R**-linear problem.

2.3. *Problem R′*

To find a doubly periodic function $\psi(z)$ analytic in D, D_k, continuous in the closure of the considered domains with the following condition

$$\psi^-(t) = \psi^+(t) + \rho_k \overline{(n(t))}^2 \overline{\psi^+(t)}, \quad t \in \partial D_k, \ k = 1, 2, \ldots, n. \tag{8}$$

Note, that the condition of double periodicity can be written in the form (compare with (7))

$$\psi(z+1) - \psi(z) = 0, \quad \psi(z+i) - \psi(z) = 0. \tag{9}$$

The **R**-linear conjugation condition (8) is obtained from (6) by differentiation along the curve ∂D_k, $\psi(z) = \varphi'(z)$ (for details see [3], p. 52).

3. Structure of the general solution

In the present section we investigate the structure of the general solutions of Problems R and R'.

THEOREM 2 *Problem R has a unique solution up to an arbitrary additive complex constant, say* $C + i\gamma$. *This solution is related to the solution* $u(z)$ *of Conductivity Problem by the formulas*

$$\varphi(z) = \begin{cases} \frac{\lambda_k+1}{2}(u(z) + iv(z)), & z \in D_k, \ k = 1, 2, \ldots, n, \\ u(z) + iv(z) \ z \in D, \end{cases} \tag{10}$$

where $v(z)$ *is a function harmonically conjugated to* $u(z)$. *The constant C is the additive arbitrary real constant from the general solution of Conductivity Problem.*

Proof is similar to the proof of the corresponding assertion on the plane [3] (p. 52). It is based on the following representations of the harmonic functions. Any function harmonic in a simply connected domain is the real part of an analytic single-valued function in this domain. Any function harmonic in a multiply connected domain is the real part of an analytic single-valued function plus logarithmic terms (for details and precise representations see [3] (p. 22) and [15]) which arise from eventual increments of the imaginary part of the analytic function along ∂D_k.

It is known (see for instance [3]) that two real relations (3)–(4) are reduced to one complex equality (6) on the plane. We give here this reduction in details on torus specifying all arising constants.

First, we prove that if $u(z)$ is any solution of Conductivity Problem, then $\varphi(z)$ introduced by (10) is a solution of Problem R. Next, we prove that if $\varphi(z)$ is any solution of Problem R, then

$$u(z) = \begin{cases} \frac{2}{\lambda_k+1} \operatorname{Re} \varphi(z), & z \in D_k, \ k = 1, 2, \ldots, n, \\ \operatorname{Re} \varphi(z), & z \in D \end{cases} \tag{11}$$

is a solution of Conductivity Problem.

Consider a solution $u(z)$ of Conductivity Problem. Let $v(z)$ be a function harmonically conjugated to $u(z)$. It is defined up to an arbitrary additive real constant. Introduce the function $\varphi(z)$ by (10) sectionally analytic in the domains D and D_k. We note that $\varphi(z)$ is continuously differentiable in the closure of the considered domains and it is single-valued in all D_k.

We now demonstrate that $\varphi(z)$ satisfies (7). The first equality (5) yields

$$\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(z+1) - \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(z) = 0.$$

Therefore,

$$\varphi'(z+1) - \varphi'(z) = 0.$$

This implies the first equality (7) with an appropriate constant d_1. Similar arguments yield the second relation (7).

In order to prove (6) introduce

$$\Delta_k = \varphi^-(t) - [\varphi^+(t) - \rho_k \overline{\varphi^+(t)}], \quad t \in \partial D_k,\ k = 1, 2, \ldots, n. \tag{12}$$

Using (10) we calculate

$$\Delta_k = u^-(t) - u^+(t) + i[v^-(t) - \lambda_k v^+(t)].$$

It follows from (3) that $\operatorname{Re}\Delta_k = 0$. The function $v(z)$ is defined in D and D_k up to arbitrary real constants, say γ and γ_k. We now prove that for any γ it is possible to introduce such γ_k that

$$v^-(t) = \lambda_k v^+(t), \quad t \in \partial D_k. \tag{13}$$

Let s denote the natural parameter of the curve ∂D_k. Applying the Cauchy–Riemann equation

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s} \tag{14}$$

to (4) we obtain

$$\frac{\partial v^-}{\partial s}(t) = \lambda_k \frac{\partial v^+}{\partial s}(t), \quad t \in \partial D_k. \tag{15}$$

Integrating (15) on s we arrive at the relation

$$v^-(t) = \lambda_k v^+(t) + c_k, \quad t \in \partial D_k, \tag{16}$$

where c_k is a constant of integration. One can take γ_k in such a way that (16) becomes (13), since the right-hand side of (16) depends additively on $\lambda_k \gamma_k + c_k$.

Therefore, $\operatorname{Im}\Delta_k = 0$ and hence $\Delta_k = 0$. The relation (6) implies that $\varphi(z)$ is single valued in D, since the right-hand side of (6) has the zero increment along each curve ∂D_k. This proves (6).

Let now $\varphi(z)$ be a solution of Problem R. Introduce $u(z)$ by (11). It is easily seen that $u(z)$ satisfies (5). Calculate the real part of (6)

$$\operatorname{Re}\varphi^-(t) = \frac{2}{\lambda_k + 1}\operatorname{Re}\varphi^+(t), \quad t \in \partial D_k. \tag{17}$$

This is equivalent to (3). Calculate the imaginary part of (6)

$$\operatorname{Im}\varphi^-(t) = \frac{2\lambda_k}{\lambda_k + 1}\operatorname{Im}\varphi^+(t), \quad t \in \partial D_k. \tag{18}$$

After differentiation on s and using the Cauchy–Riemann equation (14) we arrive at (4). Thus, we prove that $u(z)$ satisfies Conductivity Problem.

The equivalence of the problems has been established. It follows from the proof that $\varphi(z)$ depends additively on $C + i\gamma$, i.e., on one complex constant. The theorem is proved.

If $\lambda_1 = \lambda$, then $\rho = 0$ and we arrive at the $\mathbf{C}$–linear problem. In other extremal cases $\lambda_1 = 0$ (insulator inclusions) we have $\rho = -1$ and $\lambda_1 = +\infty$ (perfect conductor) $\rho = 1$. The cases $|\rho| = 1$ correspond to the Riemann–Hilbert problem. It is worth noting that in this case $\varphi(z)$ in general is multi-valued function as it was shown in [3] for the complex plane.

THEOREM 3 *General solution of the homogeneous Problem R' has the form*

$$\psi(z) = c_1\psi_1(z) + c_2\psi_2(z), \tag{19}$$

where $\psi_1(z)$ and $\psi_2(z)$ are partial linearly independent solutions of Problem R', c_1 and c_2 are arbitrary real constants.

Proof As it was already noted that differentiation of (6) and (7) implies (8) and (9), where

$$\varphi'(z) = \psi(z). \tag{20}$$

The arbitrary additive constants disappear in $\psi(z)$ after differentiation as well as the constants $c_1 + id_1$, $c_2 + id_2$ in (7). Therefore, Problems R and R' are equivalent up to an additive constant. However, there is a difference in the treatment of the constants c_1 and c_2 in these problems. According to the statement of Problem R it is assumed that c_1 and c_2 are fixed. But in the statement of Problem R' these constants are absent. Hence, the general solution of Problem R' linearly depends on two real constants c_1 and c_2 which are considered now as arbitrary constants. The partial solutions $\psi_1(z)$ and $\psi_2(z)$ correspond to solutions of Problem R with $c_1 = 1$, $c_2 = 0$ and $c_1 = 0$, $c_2 = 1$, respectively. The solutions $\psi_1(z)$ and $\psi_2(z)$ can be precisely specified without reference to Problem R. For instance, the following conditions for $\psi_1(z)$ can be added

$$\operatorname{Re}\int_z^{z+1}\psi_1(z)dz = 1, \quad \operatorname{Re}\int_z^{z+1}\psi_1(z)dz = 0. \tag{21}$$

The theorem is proved.

4. Effective conductivity tensor

The effective (macroscopic) conductivity tensor Λ is defined in the theory of homogenization [10,11] by averaging of the local laws of conductivity.

The local flux $\mathbf{q}(z)$ is defined as the vector

$$\mathbf{q}(z) = -\lambda(z)\left(\frac{\partial u}{\partial x}(z), \frac{\partial u}{\partial y}(z)\right), \tag{22}$$

where $z = x + iy$, $\lambda(z)$ is the local conductivity, i.e., $\lambda(z) = 1$ in D and $\lambda(z) = \lambda_k$ in D_k. It is convenient to represent $\mathbf{q}(z)$ as the complex value

$$q(z) = -\lambda(z)\left(\frac{\partial u}{\partial x}(z) + i\frac{\partial u}{\partial y}(z)\right). \tag{23}$$

Then (20) and (10) yield

$$q(z) = \begin{cases} -\frac{2\lambda_k}{\lambda_k+1}\overline{\psi(z)}, & z \in D_k, \\ -\overline{\psi(z)} & z \in D. \end{cases} \tag{24}$$

It follows from the proof of Theorem 3 that the partial solutions $\psi_1(z)$ and $\psi_2(z)$ of Problem R' correspond to the fluxes induced by the external gradients $(1,0) \cong 1$ and $(0,1) \cong i$. Here, we identify vectors and corresponding complex numbers.

Let $\langle\cdot\rangle$ denote the integral over the unit cell Q_0. The tensor

$$\Lambda = \begin{pmatrix} \lambda^x & \lambda^{xy} \\ \lambda^{xy} & \lambda^y \end{pmatrix} \tag{25}$$

is introduced as a coefficient in the macroscopic law which relates the external gradient with the averaged flux in the periodicity cell [10–12]

$$\langle\mathbf{q}\rangle = -\Lambda(c_1, c_2)^T, \tag{26}$$

where the vector $(c_1, c_2)^T$ denote the external gradient, $\mathbf{q}$ has the form (22). Let us write (26) in expanded form

$$\int_D (u_x, u_y)dx\,dy + \sum_{k=1}^{n}\lambda_k\int_{D_k}(u_x, u_y)dx\,dy = (c_1\lambda^x + c_2\lambda^{xy}, c_1\lambda^{xy} + c_2\lambda^y). \tag{27}$$

Using the complex form of the vectors we rewrite (27) in terms of complex potentials. At the beginning, consider the first component of (27). Applying Green's formula $\int_G u_x\,dx\,dy = \int_{\partial G} u\,dy$ for $G = D$ and $G = D_k$ to the double integrals from the first component of (27) and (3), we obtain

$$\int_{\partial Q_0} u\,dy + \sum_{k=1}^{n}(\lambda_k - 1)\int_{\partial D_k} u\,dy = c_1\lambda^x + c_2\lambda^{xy}. \tag{28}$$

Here, the equality $\partial D = \partial Q_0 - \sum_{k=1}^{n} \partial D_k$ is used. Taking into account (5) we calculate the first integral from (28) and return to doubles integrals

$$c_1 + \sum_{k=1}^{n} (\lambda_k - 1) \int_{D_k} u_x \, dx \, dy = c_1 \lambda^x + c_2 \lambda^{xy}. \tag{29}$$

Using (10) and (20) we replace u_x by ψ in (29)

$$c_1 + 2 \sum_{k=1}^{n} \rho_k \int_{D_k} \operatorname{Re} \psi(z) dx \, dy = c_1 \lambda^x + c_2 \lambda^{xy}. \tag{30}$$

Along similar lines we have from the second component of (27)

$$c_2 - 2 \sum_{k=1}^{n} \rho_k \int_{D_k} \operatorname{Im} \psi(z) dx \, dy = c_1 \lambda^{xy} + c_2 \lambda^{y}. \tag{31}$$

Multiplying (31) by i and subtracting the result from (30) we get

$$c_1(\lambda^x - i\lambda^{xy}) - ic_2(\lambda^y + i\lambda^{xy}) = c_1 - ic_2 + 2 \sum_{k=1}^{n} \rho_k \int_{D_k} \psi(z) \, dx \, dy. \tag{32}$$

Substitution of $c_1 = 1$, $c_2 = 0$ and $c_1 = 0$, $c_2 = 1$ in (32) yields the formulas

$$\lambda^x - i\lambda^{xy} = 1 + 2 \sum_{k=1}^{n} \rho_k \int_{D_k} \psi_1(z) \, dx \, dy, \tag{33}$$

$$\lambda^y + i\lambda^{xy} = 1 + 2i \sum_{k=1}^{n} \rho_k \int_{D_k} \psi_2(z) \, dx \, dy. \tag{34}$$

The invariant $I_1 = (\lambda^x + \lambda^y)/2$ of the tensor Λ has the form

$$I_1 = 1 + \sum_{k=1}^{n} \rho_k \int_{D_k} (\psi_1(z) + i\psi_2(z)) \, dx \, dy. \tag{35}$$

For macroscopically isotropic composites we have

$$\lambda^x = \lambda^y = 1 + 2 \sum_{k=1}^{n} \rho_k \int_{D_k} \psi_1(z) \, dx \, dy, \quad \lambda^{xy} = 0, \tag{36}$$

and the following equalities

$$\sum_{k=1}^{n} \rho_k \int_{D_k} \psi_2(z) dx \, dy = -i \sum_{k=1}^{n} \rho_k \int_{D_k} \psi_1(z) \, dx \, dy, \tag{37}$$

$$\operatorname{Im} \sum_{k=1}^{n} \rho_k \int_{D_k} \psi_1(z) \, dx \, dy = 0. \tag{38}$$

Equalities (37) have obvious physical interpretation

$$\left\langle \frac{\partial u_2}{\partial x} \right\rangle = -\left\langle \frac{\partial u_1}{\partial y} \right\rangle, \quad \left\langle \frac{\partial u_2}{\partial y} \right\rangle = \left\langle \frac{\partial u_1}{\partial x} \right\rangle, \tag{39}$$

i.e., in macroscopically isotropic composites the macroscopic flux does not change under rotation by about 90^0. The axis OY becomes $-OX$ under rotation, hence the sign '$-$' (minus) arises in (39). Here the potentials u_1 and u_2 correspond to the functions $\psi_1(z)$ and $\psi_2(z)$, respectively.

Consider the case when the inclusions D_k are disks $|z - a_k| < r_k$. Then application of the mean value theorem to (32) yields

$$c_1(\lambda^x - i\lambda^{xy}) - ic_2(\lambda^y + i\lambda^{xy}) = c_1 - ic_2 + 2\pi \sum_{k=1}^{n} \rho_k r_k^2 \psi(a_k). \tag{40}$$

As one can see, to determine Λ we need only $\psi(a_k)$.

5. Conclusion

In the present article the two-dimensional cell periodicity problem of the conductivity of composites has been reduced to the **R**-linear problem on torus. The equivalence of the problems is discussed in detail. The formulas (32)–(36) for the effective conductivity tensor in terms of the complex potentials have been deduced. In particular, the formula (40) has been proven. Earlier this formula with $c_1 = 1$ and $c_2 = 0$ were applied in [13,14] to compute the tensor Λ without rigorous justifications.

It could be interesting to estimate $\int_{D_k} \psi_1(z)dx\,dy$ without direct solution to Problem R'. The general theory of bounds for Λ is presented by Milton [12] without an address to the **R**-linear problem.

References

[1] Gakhov, F.D., 1977, *Boundary Value Problems* (Moscow: Nauka).
[2] Mikhajlov, L.G., 1970, *A New Class of Singular Integral Equations* (Groningen: Wolters-Noordhoff Publ.).
[3] Mityushev, V.V. and Rogosin, S.V., 2000, *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions Theory* (Boca Raton: Chapman & Hall/CRC).
[4] Wen, G.C. and Begehr, H., 1990, *Boundary Value Problems for Elliptic Equations and Systems*, *Pitmans Monographs* (New York: John Wiley & Sons).
[5] Dzhuraev, A., 1992, *Method of Singular Integral Equations*, *Pitmans Monographs* (New York: John Wiley & Sons).
[6] Komyak, I.I., 1977, Closed form solution to a two-dimensional singular integral equation with analytic coefficients. *Doklady AN BSSR*, **21**(7), 583–586 [Russian].
[7] Komyak, I.I., 1980, On two-dimensional singular integral equations with analytic coefficients. *Differencialnye Uravneniya*, **16**(5), 908–916 [Russian].
[8] Litvinchuk, G.S. and Spitkovsky, I.M., 1987, *Factorization of Measurable Matrix Functions*, *Operator Theory: Advances and Applications*, Vol. 25 (Basel: Birkhäuser Verlag).
[9] Gatalskaya, T.I., 2003, Explicit solution of the Riemann boundary value problem for double periodic functions in case of compound contour. *Mathematical Modeling and Analysis*, **8**(1), 13–24.
[10] Bakhvalov, N. and Panasenko, G., 1989, *Mathematical Problems in the Mechanics of Composite Materials* (Dordrecht: Kluwer Academic Publishers).

[11] Jikov, V.V., Kozlov, S.M. and Olejnik, O.A., 1994, *Homogenization of Differential Operators and Integral Functionals* (Berlin: Springer-Verlag).
[12] Milton, G.W., 2002, *Mechanics of Composites* (Cambridge: Cambridge University Press).
[13] Berlyand, L. and Mityushev, V., 2001, Generalized Clausius-Mossotti formula for random composite with circular fibers. *Journal of Statistical Physics*, **102**(1/2), 115–145.
[14] Mityushev, V., 2001, Transport properties of doubly periodic array of circular cylinder and optimal design problem. *Applied Mathematics and Optimization*, **44**, 17–31.
[15] Mityushev, V.V. and Adler, P.M., 2002, Longitudial permeability of a doubly periodic rectangular array of circular cylinders I. *Zeitschrisft for Angewandte Mathematic und Mechanik*, **82**(5), 335–345.