Transport properties of two-dimensional composite materials with circular inclusions

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We consider the transport properties of a two-dimensional, two-component composite medium made from a collection of non-overlapping, identical circular disks, imbedded in an otherwise uniform host. Both components are isotropic conductors, but the position of the inclusions is arbitrary. The study is based on the analytic properties of such composite materials described by Bergman (1978, 1982, 1985), Bergman & Dunn (1992), Milton (1981) and the homogenization theory of random media advanced by Golden & Papanicolaou (1983), Jikov et al. (1994). The crucial point of our study is application of the method of functional equations.

1 Introduction

The transport properties of two-dimensional, two-component composite medium made from a collection of non-overlapping, identical circular disks, imbedded in an otherwise uniform host are considered. A number of workers have followed by different approach, by assuming a specific regular geometry for the composite material. In this way, they have been inspired by a classic paper of Lord Rayleigh (1892). McPhedran (1986), McPhedran & Milton (1987), McPhedran et al. (1988), Sangani & Yao (1988) obtained an infinite systems of linear algebraic equations for the multipole coefficients. These systems had been truncated to give various low-order formulae for the effective transport properties. Mityushev (1995a, 1995b, 1996, 1997a, 1997b) obtained exact analytic formulae for arbitrary regular arrays of circular disks by using the method of functional equations.

Bergman (1978, 1982, 1985), Bergman & Dunn (1992) and Milton (1981) discussed analytical properties of macroscopic moduli of general two-phase composite materials as a function of the conductivity ratio. They also formulated bounds on the effective constant from available information. (For a recent review of the theory of bounds see Clark & Milton (1995)). Golden & Papanicolaou (1983) and Jikov et al. (1994) extended the Bergman-Milton theory to random media. They proposed a rigorous mathematical theory of the homogenization of elliptic operators with random coefficients. We shall call this theory by the homogenization theory of random media. A presentation of the problem of estimating the effective transport properties of two-phase random media is given by Markov & Zvyatkov (1991) and Torquato (1991).

In the present paper we consider a composite material containing infinite parallel cylindrical inclusions (identical disks in the two-dimensional statement) randomly embedded in a homogeneous matrix. We evaluate the effective conductivity tensor Λ_e in the framework of the homogenization theory of random media. Applying the method of functional equations we deduce a simple algorithm for approximate analytic formulae for Λ_e. These formulae allow us to discuss some particular problems. For instance, a distribution of disks on plane is given, and one asks the following natural question. Is this isotropic or anisotropic material in macroscale? If the material is anisotropic then it is interesting to determine the principal axes. This question is answered in Sec.5 by using the concept of a generalized Rayleigh's sum.

2 Formulation of the problem

We consider a random composite material in the framework of the homogenization theory of random media by Golden&Papanicolaou (1983) and Jikov et al. (1994). Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\lambda(z,\omega)$ be a strictly stationary random field of the complex variable $z = x + iy$, $\omega \in \Omega$, deriving a random two-dimensional composite material with identical circular inclusions of the radius r. The inclusions have the scalar conductivity λ_1 and are separated by matrix of unit conductivity. More specifically, each event $\omega \in \Omega$ corresponds to a realization of the composite material, i.e. to a set of the non-overlapping disks on the complex plane $\mathbb{C}$. For each fixed ω the function $\lambda(z,\omega)$ takes only two values: λ_1 in the inclusion and the unity in the matrix. Let us consider the following boundary value problem. Find stationary random potential $u(z,\omega)$ such that

$$\nabla\left(\lambda(z,\omega)\nabla u(z,\omega)\right) = 0,\ z \in \mathbb{C},\ \omega \in \Omega,\ \int_\Omega P(d\omega)\nabla u(z,\omega) = \mathbf{e}, \quad (2.1)$$

where $\nabla := (\partial/\partial x, \partial/\partial y)$. The constant vector $\mathbf{e} = e_1 + ie_2$ is given. The function $u(z,\omega) - Re\mathbf{e}z$ is bounded at infinity. The effective properties tensor

$$\Lambda_e = \begin{pmatrix} \lambda_e^x & \lambda_e^{xy} \\ \lambda_e^{xy} & \lambda_e^y \end{pmatrix}$$

is defined by the relation

$$\Lambda_e \mathbf{e} = \int_\Omega P(d\omega)\,\lambda(z,\omega)\,\nabla u(z,\omega). \tag{2.2}$$

According to the homogenization theory of random media the problem (2.1) has a unique solution up an additive constant. Moreover, for almost $\omega \in \Omega$ the strictly stationary random field $\lambda(z,\omega)$ admits homogenization, and the homogenized tensor Λ_e is independent of ω. The last result allows us to take an element $\omega \in \Omega$ corresponding to a typical distribution of inclusions and calculate Λ_e in this particular case. Since the effective properties tensor Λ_e is independent of ω, hence we get the value Λ_e for whole class Ω of the composite materials under consideration.

So let us take a typical distribution of the inclusions represented by a set of mutually disjoint discs $D_k := \{z \in \mathbb{C} : |z - a_k| < r\}$ $(k = 0,1,...)$, where $0 = |a_0| < |a_1| \leq |a_2| \leq ...$. Let $D := \mathbb{C}\backslash(H \cup \partial H)$, where $H := \cup_{k=1}^{\infty} D_k$, ∂H isthe boundary of H. We study conductivity of the composite material, when the domains D and D_k $(k = 0,1,...)$ are occupied by materials of conductivity $\lambda = 1$ and λ_1, respectively (see Fig.1). We find the potentials $u(z)$ and $u_k(z)$ harmonic in D and D_k $(k = 0,1,...)$ with the following boundary conditions:

$$u = u_k, \ \frac{\partial u}{\partial n} = \lambda_1 \frac{\partial u_k}{\partial n} \text{ on the circumferences } |t - a_k| = r, \ k = 0,1,... , \tag{2.3}$$

where $\partial/\partial n$ is the outward normal derivative. Moreover, $u(z,\omega) - Re\,\mathbf{e}z$ satisfies the property $\mathcal{P}^0$. We say that a function $f(z)$ satisfies the property $\mathcal{P}^0$ if it is continuous in $(D \cup \partial D) \cap U_R$ for each $R > 0$, where $U_R := \{z \in \mathbb{C} : |z| < R\}$ and bounded in $D \cup \partial D$. The function $f(z)$ satisfies the property $\mathcal{P}^1$ if it is continuously differentiable in $(D \cup \partial D) \cap U_R$ for each $R > 0$ and satisfies the property $\mathcal{P}^0$ with $|\nabla f(z)|$.

In accordance with the homogenization theory of random media the definition of the effective properties tensor Λ_e (2.2) is consistent with the definition of the ensemble averaging

$$\Lambda_e \mathbf{e} = \lim_{n \to \infty} |G_n|^{-1} \left[\int\int_{F_n} \nabla u dxdy + \lambda_1 \sum_{k=0}^{n} \int\int_{D_k} \nabla u_k dxdy \right]. \tag{2.4}$$

Here G_n is a simply connected bounded domain containing $D_0, D_1, ..., D_n,\ |G_n|$ is the area of $G_n,\ F_n := G_n \setminus \cup_{k=0}^{n} (D_k \cup \partial D_k),\ \lim_{n\to\infty} F_n = D$. In order to find Λ_e it is sufficiently to apply (2.4) for $\mathbf{e} = 1$ and $\mathbf{e} = i$. For the definiteness we take only the external field applied in the x-direction. Then

$$\lambda_e^x - i\lambda_e^{xy} = \tag{2.5}$$

$$\lim_{n\to\infty} |G_n|^{-1} \left[\int\int_{F_n} (u_x - iu_y)\, dxdy + \lambda_1 \sum_{k=0}^{n} \int\int_{D_k} \left((u_k)_x - i\,(u_k)_y\right) dxdy\right].$$

The problem (2.3) is equivalent to the following $\mathbb{R}$-linear boundary value problem (see, for instance Mityushev (1996, 1997b))

$$\phi(t) = \phi_k(t) - \rho\overline{\phi_k(t)} - t, |t - a_k| = r,\ k = 0, 1, ...\ , \tag{2.6}$$

where $\rho := (\lambda_1 - 1)/(\lambda_1 + 1)$ is a Bergman's parameter. The unknown functions $\phi(z)$ and $\phi_k(z)$ are analytic in D and D_k, respectively. Moreover, the function $\phi(z)$ satisfies the property $\mathcal{P}^1$. The harmonic and analytic functions are related by the identities

$$u(z) = Re\ (\phi(z) + z),\ u_k(z) = \frac{2}{\lambda_1 + 1} Re\, \phi_k(z).$$

Here and after we write z when we consider a relation in a domain, and t - in a contour.

Let us transform the relation (2.5). Applying Green's formula we arrive at the relation

$$\lambda_e^x - i\lambda_e^{xy} = \lim_{n\to\infty} |G_n|^{-1} \left[i \int_{\partial G_n} u d\overline{z} + 2\rho \sum_{k=0}^{n} \int\int_{D_k} \phi_k^{/}(z)\, dxdy\right], \tag{2.7}$$

where $\phi_k^{/}(z) = \frac{\lambda_1+1}{2}\left((u_k)_x - i\,(u_k)_y\right)$. We have

$$\begin{aligned} \lim_{n\to\infty} |G_n|^{-1}\, i \int_{\partial G_n} u d\overline{z} &= \\ \lim_{n\to\infty} |G_n|^{-1} \left[\int_{\partial G_n} ixdx + xdy + \int_{\partial G_n} (bounded\ term)\, d\overline{z}\right] &= 1, \end{aligned}$$

since $u(z) = x + bounded\ term$ as $z \to \infty$. By virtue of the mean value theorem of harmonic functions we have

$$\int\int_{D_k} \phi_k^{/}(z)\, dxdy = \pi r^2 \phi_k^{/}(a_k).$$

Therefore, (2.7) implies

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho v \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \phi_k^{/}(a_k), \tag{2.8}$$

where $v := \lim_{n\to\infty} (n+1)\pi r^2 |G_n|^{-1}$ is the area fraction of the inclusions. Without loss of generality we assume that the average number of inclusions per unit area $\lim_{n\to\infty}(n+1)|G_n|^{-1} = 1$. Then $v = \pi r^2$.

3 Method of functional equations

Let us introduce the Banach space $\mathcal{B}$ consisting of functions analytic in all discs D_k, continuous in $D_k \cup \partial D_k$ $(k = 0, 1, ...,)$ and bounded in $H \cup \partial H$ with the norm $\|\Psi\| := \sup_k \max_{D_k\cup\partial D_k} |\psi_k(z)|$, where $\Psi(z) = \psi_k(z)$ in $D_k \cup \partial D_k$. Convergence in $\mathcal{B}$ means almost uniform convergence, i.e. uniform convergence in each compact subset of H.

Lemma 3.1. *Let the function $\psi_m(z)$ is analytic in D_m and continuous in $D_m \cup \partial D_m$ $(m = 0, 1, ...,)$. Let $z_m^* := r^2/\overline{(z - a_m)} + a_m$ be the inversion with respect to $|t - a_m| = r_m$. Then the series*

$$S\Psi(z) := \sum_{m=0}^{\infty} (A_m\psi_m)(z), \tag{3.1}$$

where

$$\begin{aligned} (A_m\psi_m)(z) &: = (z - a_m)^{-2}\,\overline{\psi_m(z_m^*)} - \Delta_m\overline{\psi_m(a_m)}\ \ (m = 0, 1, ...,), \\ \Delta_0 &: = 0,\ \Delta_m := a_m^{-2}\ \ (m = 1, 2, ...,) \end{aligned}$$

converges absolutely and almost uniformly in D. The function $S\Psi(z)$ is analytic in D and satisfies the property $\mathcal{P}^0$.

P r o o f. Following Mityushev (1997b) we fix a compact subset $K \subset\subset D$ and consider the series

$$\sum_{m=N}^{\infty} (A_m\psi_m)(z), \tag{3.2}$$

where N is chosen in such a way that $|z - a_m| \geq h > r^2/(r - \varepsilon)$ for sufficiently small $\varepsilon > 0$ and for all $m \geq N$ and $z \in K$. Let

$$\psi_m(z) = \sum_{l=0}^{\infty} \psi_{ml}(z - a_m)^l$$

be the Taylor expansion of $\psi_m(z)$. Then

$$\sum_{m=N}^{\infty} (A_m\psi_m)(z) = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 := \sum_{m=N}^{\infty} \overline{\psi_{m0}} \left[(z-a_m)^{-2} - a_m^{-2}\right],\ \Sigma_2 := \sum_{m=N}^{\infty}\sum_{l=1}^{\infty} \overline{\psi_{ml}} r^{2l} (z-a_m)^{-l-2}.$$

We have

$$|\Sigma_1| \leq T \sum_{m=N}^{\infty} \left|(z-a_m)^{-2} - a_m^{-2}\right|, \tag{3.3}$$

where $T := \sup_m |\psi_{m0}| = \sup_m |\psi_m(a_m)|$. It follows from Lemma A.2 that the series (3.3) converges uniformly in K. The inequality $|z-a_m|^{-l+1} \leq h^{-l+1}$ $(l = 1, 2, ...)$ implies

$$\left|\sum_{m=N}^{\infty}\sum_{l=1}^{\infty} \overline{\psi_{ml}} r^{2l} (z-a_m)^{-3} (z-a_m)^{-l+1}\right| \leq h \sum_{m=N}^{\infty}\sum_{l=1}^{\infty} |\psi_{ml}| r^{2l} |z-a_m|^{-3} h^{-l} \leq$$

$$hMc \sum_{l=1}^{\infty} \left[r^2 (r-\varepsilon)^{-1} h^{-1}\right]^l < +\infty,$$

where $M := \max_{z\in K} \sum_{m=N}^{\infty} |z-a_m|^{-3}$ (see Lemma A.3). Here Cauchy's inequality $|\psi_{ml}| \leq c(r-\varepsilon)^{-l}$ $(l = 1, 2, ...)$ is used. Thus, we have proved that the series (3.2) converges uniformly in K. Therefore, the series (3.1) converges almost uniformly in D. The properties of the function $S\Psi(z)$ follow from the properties of the uniformly convergent series of analytic functions.

The lemma is proved.

Differentiating (2.6) we arrive at the following boundary value problem

$$\psi(t) = \psi_k(t) + \rho r^2 (t-a_k)^{-2} \overline{\psi_k(t)} - 1,\ |t-a_k| = r,\ k = 0, 1, \ldots, \tag{3.4}$$

where $\psi(z) := \phi^{/}(z)$ and $\psi_k(z) := \phi_k^{/}(z)$. Here we apply the relation (see, for instance Mityushev (1996))

$$\left(\overline{\phi_k(t)}\right)^{/} = \left(\overline{\phi_k(t_k^*)}\right)^{/} = -r^2 (t-a_k)^{-2}\, \overline{\phi_k^{/}(t)},\ \ |t-a_k| = r.$$

Let us introduce the function

$$\Phi(z) := \psi_k(z) - \rho r^2 \sum_{m\neq k} (A_m\psi_m)(z) + \rho r^2 \Delta_k \overline{\psi_k(a_k)} - 1,\ |z-a_k| \leq r,\ k = 0, 1, \ldots,$$

$$\Phi(z) := \psi(z) - \rho r^2 \sum_{m=0}^{\infty} (A_m \psi_m)(z), \; z \in D,$$

where the sum $\sum_{m \neq k}$ contains the terms with $m = 0, 1, \ldots$, except $m = k$. It follows from Lemma 3.1 that the function $\Phi(z)$ is analytic in D and D_k $(k = 0, 1, \ldots)$, continuous in all $D_k \cup \partial D_k$ $(k = 0, 1, \ldots)$ and satisfies the property $\mathcal{P}^0$. We now proceed to calculate the jump of $\Phi(z)$ along $|t - a_k| = r$:

$$J_k := \lim_{z \to t, z \in D} \Phi(z) - \lim_{z \to t, z \in D_k} \Phi(z) =$$

$$\psi(t) - \rho r^2 \left[(t - a_k)^{-2} \overline{\psi_k(t)} - \Delta_k \overline{\psi_k(a_k)} \right] - \psi_k(t) - \rho r^2 \Delta_k \overline{\psi_k(a_k)} + 1, \; k = 0, 1, \ldots .$$

Taking into account (3.4) we obtain $J_k = 0$. Using the theorem of analytic continuation and the Liouville theorem we conclude that $\Phi(z) \equiv q =$ *constant*. From the definition of $\Phi(z)$ we obtain the following system of functional equations

$$\psi_k(z) = \rho r^2 \sum_{m \neq k} \left[(z - a_m)^{-2} \overline{\psi_m(z_m^*)} - \Delta_m \overline{\psi_m(a_m)} \right] - \rho r^2 \Delta_k \overline{\psi_k(a_k)} + 1 + q,$$

$$|z - a_k| \leq r, \; k = 0, 1, \ldots \tag{3.5}$$

with respect to the functions $\psi_k(z)$ analytic in $|z - a_k| < r$ and continuous in $|z - a_k| \leq r$. Moreover, there exist $\sup_k \max_z |\psi_k(z)|$, where $|z - a_k| \leq r$ in the $\max_z$. The system (3.5) can be considered as the equation

$$\Psi = \rho r^2 \mathbf{A} \Psi + 1 + q \tag{3.6}$$

in the space $\mathcal{B}$. Here $\Psi(z) = \psi_k(z)$ in $|z - a_k| \leq r$, $\mathbf{A}\Psi(z) := \sum_{m \neq k} (A_m \psi_m)(z) - \Delta_k \overline{\psi_k(a_k)}$ in $|z - a_k| \leq r$.

Lemma 3.2. *The operator* $\mathbf{A}$ *is compact in* $\mathcal{B}$.

P r o o f. The operators

$$\sum_{m=0}^{N} (A_m \psi_m)(z), \; N = 1, 2, \ldots \tag{3.7}$$

are compact in $\mathcal{B}$ (see Mityushev (1997b)). The operator $\Delta_k \overline{\psi_k(a_k)}$ is compact as a degenerated operator. Lemma 3.1 implies that $\mathbf{A}$ is a compact operator as the limit of the compact operators (3.7) in the space $\mathcal{B}$.

The lemma is proved.

Let us denote by $\mathbf{A}^k$ the k-th degree of the operator $\mathbf{A}$.

Theorem 3.1. *Equation (3.6) has the unique solution*

$$\Psi = (1+q)\sum_{k=0}^{\infty}\left(\rho r^2\right)^k \mathbf{A}^k 1$$

for each fixed q. The last series converges in $\mathcal{B}$.

P r o o f. In order to prove the theorem we show that $\rho r^2 R_{\mathbf{A}} < 1$, where $R_{\mathbf{A}}$ is the spectral radius of the operator $\mathbf{A}$. It is sufficient to show the inequality $r^2 R_{\mathbf{A}} \leq 1$, since $|\rho| < 1$. The spectrum of the operator $\mathbf{A}$ consists only of eigenvalues. The inequality $r^2 R_{\mathbf{A}} \leq 1$ is valid if and only if there exist a complex constant ν such that $|\nu| < 1$ and the equation

$$\Psi = \nu r^2 \mathbf{A}\Psi \tag{3.8}$$

has only the zero solution. If $\Psi(z) = \psi_k(z)$ in $|z - a_k| \leq r$ is a solution of (3.8) then the function

$$\psi(z) = \nu r^2 \sum_{m=0}^{\infty}\left(A_m \psi_m\right)(z)$$

is analytic in D and satisfies the property $\mathcal{P}^0$. Using (3.8) we arrive at the $\mathbb{R}$-linear problem

$$\psi(t) = \psi_k(t) + \nu r^2 (t - a_k)^{-2}\,\overline{\psi_k(t)},\ |t - a_k| = r,\ k = 0, 1, ... \,. \tag{3.9}$$

The problem (3.9) has only the zero solution since $|\nu| < 1$ (see Appendix B).

The theorem is proved.

We shall find $\psi_k(z)$ from the functional equations (3.5) in the form

$$\psi_k(z,\rho) = \psi_k^0(z) + \rho\psi_k^1(z) + \rho^2\psi_k^2(z) + ... \,. \tag{3.10}$$

Also, we consider the constant q as an analytic function on ρ :

$$q(\rho) = q_0 + q_1\rho + q_2\rho^2 ... \,. \tag{3.11}$$

By substituting (3.10), (3.11) into (3.5) and collecting terms with ρ^m we obtain the following recurrent formulae

$$\psi_k^0(z) = 1 + q_0, \tag{3.12}$$

$$\psi_k^l(z) = r^2 \sum_{m\neq k}\left[(z - a_m)^{-2}\,\overline{\psi_m^{l-1}(z_m^*)} - \Delta_m \overline{\psi_m^{l-1}(a_m)}\right] - r^2 \Delta_k \overline{\psi_k^{l-1}(a_k)} + q_l,$$

$$|z - a_k| \leq r,\ k = 0, 1, 2, ...;\ l = 1, 2, ...$$

4 Method of perturbation

In order to apply Theorem 3.1 and the scheme (3.12) to calculate the tensor Λ_e we must determine the constant q appearing in the definition of $\Phi(z)$. We shall do it by a method of perturbation. This method is concluded in finding a solution of the problem (2.6) in the form of the expansions

$$\phi(z,\rho) = \phi^0(z) + \rho\phi^1(z) + \rho^2\phi^2(z) + ..., \tag{4.1}$$

$$\phi_k(z,\rho) = \phi_k^0(z) + \rho\phi_k^1(z) + \rho^2\phi_k^2(z) + ...$$

with respect to ρ. By substituting these expansions into the boundary condition (2.6) and collecting terms with ρ^m we obtain a cascade of the problems. The zero one is

$$\phi^0(t) = \phi_k^0(t) - t,\ |t-a_k| = r,\ k = 0,1,2,... \ .$$

The first problem is

$$\phi^1(t) = \phi_k^1(t) - \overline{\phi_k^0(t)},\ |t-a_k| = r,\ k = 0,1,2,... \ .$$

and so on. On the l-th step we have

$$\phi^l(t) = \phi_k^l(t) - \overline{\phi_k^{l-1}(t)},\ |t-a_k| = r,\ k = 0,1,2,... \ , l = 1,2,... \ . \tag{4.2}$$

Since $\phi_k^0(z) = z$ is the solution of the zero problem, hence the first problem becomes

$$\phi^1(t) = \phi_k^1(t) - \left(r^2/(t-a_k) + \overline{a_k}\right),\ |t-a_k| = r,\ k = 0,1,2,... \ . \tag{4.3}$$

The last equalities mean that $\phi^1(z)$ is analytically continued to all $(D_k \cup \partial D_k) \setminus \{a_k\}$ and has the principle part $-r^2/(z-a_k)$ at a_k. It follows from the theory of meromorphic functions (see Appendix A) that

$$\phi^1(z) = -r^2\left(F_1(z) - G_1(z)\right),$$

where

$$F_1(z) = \frac{1}{z} + \sum_{m=1}^{\infty}\left(\frac{1}{z-a_m} + \frac{1}{a_m} + \frac{z}{a_m^2}\right), \tag{4.4}$$

$G_1(z)$ is an entire function. The series (4.4) converges absolutely and almost uniformly in $\mathbb{C}\setminus \cup_{m=0}^{\infty}\{a_m\}$. From (4.3) and (4.4) we have

$$\phi_0^1(z) = -r^2\left[\sum_{m=1}^{\infty}\left(\frac{1}{z-a_m} + \frac{1}{a_m} + \frac{z}{a_m^2}\right) - G_1(z)\right], k = 0,$$

$$\phi_k^1(z) = -r^2\left[\frac{1}{z} + \sum_{m=1\, m\neq k}^{\infty}\left(\frac{1}{z-a_m} + \frac{1}{a_m} + \frac{z}{a_m^2}\right) + \frac{1}{a_k} + \frac{z}{a_k^2} - G_1(z)\right] + \overline{a_k},\ k = 1,2,..., \tag{4.5}$$

Let us compare this result with the result obtained by the method of functional equations . Let us note that the uniqueness theorem of the analytic function theory implies the relation

$$\left(\phi_k^1(z)\right)' = \psi_k^1(z),$$

where $\psi_k^1(z)$ is taken from (3.10). Using (3.12) with $l = 1$ we calculate

$$\psi_k^1(z) = r^2(1+q_0)\left\{\sum_{m\neq k}\left[(z-a_m)^{-2} - \Delta_m\right] - \Delta_k\right\} + q_1,$$

where the sum $\sum_{m\neq k}$ contains the terms with $m = 0,1,...;\ m \neq k$. Differentiating (4.5) we obtain

$$\left(\phi_k^1(z)\right)' = r^2\left\{\sum_{m\neq k}\left[(z-a_m)^{-2} - \Delta_m\right] + G_1'(z) - \Delta_k\right\}.$$

Comparing the last two equalities we have

$$q_0 = 0,\ q_1 = r^2 G_1'(z).$$

Hence, the function $G_1(z)$ has the form

$$G_1(z) = r^{-2}q_1 z + c_1, \tag{4.6}$$

where c_1 is a constant, which does not impact on the effective conductivity tensor. So we have to determine only $r^{-2}q_1$.

The function $F_1(z)$ at infinity has to be compensated by the linear function $G_1(z)$, since the function $\phi^1(z)$ is bounded at infinity. Let us fix the point w in D. Let γ be a smooth simple curve connected the points $z = w$ and $z = \infty$, $\gamma \subset D \cup \partial D$ (see Fig.1). Let us calculate the jump of $F_1(z)$ along γ from w to $w + \Delta z$:

$$f(\gamma, \Delta z) := F_1(w+\Delta z) - F_1(w) = \Delta z\left[-\frac{1}{w(w+\Delta z)} - \sum_{k=1}^{\infty}\left(\frac{1}{(w-a_k)(w-a_k+\Delta z)} - \frac{1}{a_k^2}\right)\right].$$

We have

$$q_1 = r^2\lim_{\Delta z\to\infty}(\Delta z)^{-1} f(\gamma,\Delta z) = r^2\lim_{\Delta z\to\infty}\sum_{k=1}^{\infty}\left(\frac{1}{a_k^2} - \frac{1}{(w-a_k)(w-a_k+\Delta z)}\right), \tag{4.7}$$

and

$$\psi_k^1(z) = r^2 \sum_{m \neq k} \left[(z - a_m)^{-2} - \Delta_m \right] - r^2 \Delta_k + q_1. \tag{4.8}$$

We have determined $\psi_k^1(z)$ and q_1. We now proceed to determine $\psi_k^2(z)$ and q_2 by the same method.

It follows from (3.12) with $l = 2$ that

$$\psi_k^2(z) = r^2 \sum_{m \neq k} \left[(z - a_m)^{-2} \overline{\psi_m^1(z_m^*)} - \Delta_m \overline{\psi_m^1(a_m)} \right] - r^2 \Delta_k \overline{\psi_k^1(a_k)} + q_2, \tag{4.9}$$

where $\psi_m^1(z)$ has the form (4.8), q_2 is an undetermined constant. The relation (4.2) with $l = 2$ implies

$$\phi^2(t) = \phi_k^2(t) - \overline{\phi_k^1(t)}, \; |t - a_k| = r, \; k = 0, 1, 2, ... \,. \tag{4.10}$$

One can consider (4.10) as a boundary value problem with respect to $\phi^2(z)$ and $\phi_k^2(z)$ with the functions $\overline{\phi_k^1(t)}$ defined by (4.5) with $G_1(z) = q_1 z + c_1$. Let us note that the function $\overline{\phi_k^1(z_k^*)}$ is meromorphic in $|z - a_k| < r$. Hence, the equality (4.10) implies

$$\phi^2(z) = F_2(z) + G_2(z),$$

where $F_2(z)$ is a meromorphic function, $G_2(z)$ is an entire function. Then

$$\left(\phi_k^2(z)\right)^{/} = F_2^{/}(z) + \left[\overline{\phi_k^1(z_k^*)} \right]^{/} + G_2^{/}(z).$$

The constant q_2 can be calculated by $F_2(z)$ and $G_2(z)$.

By the same method we can find $\psi_k^3(z)$ and q_3, $\psi_k^4(z)$ and q_4 and so on.

5 Effective conductivity tensor

In Sec.3 and 4 we determine the functions $\psi_k(z) = \phi_k^{/}(z)$. Using (2.8) we now proceed to calculate the value

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho v \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \psi_k(a_k). \tag{5.1}$$

It follows from (3.10) and the equality $\psi_k^0(z) = 1$ that (5.1) can be written in the form

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho v + 2\rho^2 v Q_1 + 2\rho^3 v Q_2 ..., \tag{5.2}$$

where $Q_l := \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \psi_k^l (a_k)\,,\ \ l = 1, 2, \ldots$. Each limit Q_l exists because the value $\lambda_e^x - i\lambda_e^{xy}$ is an analytic function with respect to ρ in the unit disk $|\rho| < 1$ (see Bergman (1978, 1982, 1985), Bergman & Dunn (1992)). It follows from (5.2) that

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho v + 2\rho^2 v Q_1 + o(\rho^2),\ \text{as}\ \rho \to 0, \tag{5.3}$$

We now proceed to calculate the limit

$$Q_1 = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \psi_k^1 (a_k)\,.$$

Using (4.8), (4.7) we obtain $Q_1 = r^2 S_2$, where

$$S_2 := \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{m=1\ m\neq k}^{\infty} \left[(a_k - a_m)^{-2} - \Delta_m\right] + \lim_{\Delta z\to\infty} \sum_{k=1}^{\infty} \left(\frac{1}{a_k^2} - \frac{1}{(w - a_k)(w - a_k + \Delta z)}\right). \tag{5.4}$$

We call S_2 the generalized Rayleigh sum of second order. It depends only on geometric parameters of the composite material. Using S_2 one can write (5.3) in the form

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho v + 2\rho^2 v^2 S_2/\pi + o(\rho^2),\ \text{as}\ \rho \to 0, \tag{5.5}$$

If the points a_k generate a doubly periodic lattice, then

$$S_2 = \alpha^{-1} 2\zeta\left(\alpha/2\right). \tag{5.6}$$

Here the values $\alpha > 0$ and $\beta \in \mathbb{C}$ $(Im\,\beta = \alpha^{-1})$ are the fundamental vectors on the complex plane $\mathbb{C}$ generating the lattice, $\zeta(z)$ is the Weierstrass function (see Hurwitz (1964)). The value S_2 from (5.6) is related to the conditionally convergent sum $\sum_{k=1}^{n} a_k^{-2}$ discussed by Rayleigh (1892), McPhedran et al. (1978), Perrins et al. (1979), Mityushev (1995a, 1997c) and others.

It follows from the homogenization theory of random media that the limits (4.7) and (5.4) exist and do not depend on w and γ. This notation can be useful to discuss the homogenization of a composite material. If the limit (4.7) depends on w then the material in question is not homogenized. This situation is possible when the area fraction of inclusion v does not exist. If the limit (4.7) depends on γ then we have different properties of the material in different directions asymptotically defined by the curve γ (see Fig.1).

In order to determine λ_e^y we have to apply the external field along the y axis. In this case we arrive at the $\mathbb{R}$-linear problem

$$\phi(t) = \phi_k(t) - \rho\overline{\phi_k(t)} + it, |t - a_k| = r,\ k = 0, 1, \ldots\,, \tag{5.7}$$

instead of (2.6). This problem is solved by the same method as the problem (2.6). In particular we have

$$\lambda_e^y = 1 + 2\rho v + 2\rho^2 v^2 \left(2 - Re\, S_2/\pi\right) + o(\rho^2), \text{ as } \rho \to 0. \tag{5.8}$$

The relations (5.5) and (5.8) determine the effective conductivity tensor Λ_e in the second order approximation with respect to ρ. Rather than presenting Λ_e by (5.5) and (5.8), it is more useful to give the components of Λ_e along its major and minor axes, λ_{maj} and $\lambda_{\min}$, and the angle between one of the principal axes and the x axis, θ. Thus the result of the computations are presented as follow

$$\lambda_{maj} = 1 + 2\rho v + 2\rho^2 v^2 + 2\rho^2 v^2 \left|S_2/\pi - 1\right|, \tag{5.9}$$

$$\lambda_{\min} = 1 + 2\rho v + 2\rho^2 v^2 - 2\rho^2 v^2 \left|S_2/\pi - 1\right|, \ \tan 2\theta = \frac{1}{2} \arg\left(S_2/\pi - 1\right),$$

where $\arg\left(S_2/\pi - 1\right)$ is the argument of the complex number $S_2/\pi - 1$.

Using Keller's theorem for random media (see Berdichevskij (1983)) one can derive (5.8) and (5.9) directly from (5.5).

6 Conclusion

We have studied the transport properties of a two-dimensional, two-component composite medium made from a collection of non-overlapping, identical circular disks, imbedded in an otherwise uniform host by the method of functional equations. An algorithm to calculate the effective conductivity tensor in analytic form has been proposed in Sec.3 and 4. This tensor has been evaluated to within a second order approximation in Sec.5. We have also discussed a generalized Rayleigh's sum of second order and its both the applications to homogenization and anisotropy of random media.

7 Acnowledgement

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Appendix A

The present section is based on the theories of the meromorphic and elliptic functions (see, for instance, Hurwitz (1964)).

Lemma A.1. *Let the set* $A := \{a_k\}_{k=0}^{\infty} \subset \mathbb{C}$ *satisfies the conditions of Sec.2, i.e. the discs* $|z - a_k| < r$ *model inclusions of a homogenized composite material. Then the series*

$$\sum_{k=1}^{\infty} |a_k|^{-m}$$

converges for $m = 3$ *and diverges for* $m = 2$.

P r o o f. Let us construct the lattice $\mathcal{Q}_s$ generated by the fundamental vectors s and is with the zero cell $\{z = x + iy \in \mathbb{C} : |x| < s/2, |y| < s/2\}$. Let e_j be the center of the j-th cell of $\mathcal{Q}_s$. We assume that $e_0 = 0$, $|e_1| \leq |e_2| \leq |e_3| \leq \dots$.

If $m = 3$ then we take a lattice $\mathcal{Q}_s$ where s is chosen in such a way that each cell contains no more than one point of A. Let $b_j = e_j$ if the j-th cell does not contain points of A, and $b_j = a_k$ in the opposite case. Here the points b_j and a_k belong to the same cell. It is known that the series $\sum_{j=1}^{\infty} |e_j|^{-3}$ converges. Let us prove that the series $\sum_{j=1}^{\infty} |b_j|^{-3}$ is convergent too.

Using the inequality $||b_j| - |e_j|| \leq s/\sqrt{2}$ we have

$$\sum_{j=1}^{\infty} \left(|b_j|^{-3} - |e_j|^{-3}\right) \leq \frac{s}{\sqrt{2}} \sum_{j=1}^{\infty} \left(|b_j|^{-3} |e_j|^{-1} + |b_j|^{-2} |e_j|^{-2} + |b_j|^{-1} |e_j|^{-3}\right).$$

Estimate the first term:

$$\sum_{j=1}^{\infty} \left(|b_j|^{-3} |e_j|^{-1} - |e_j|^{-4}\right) \leq \frac{s}{\sqrt{2}} \sum_{j=1}^{\infty} \left(|b_j|^{-3} |e_j|^{-2} + |b_j|^{-2} |e_j|^{-3} + |b_j|^{-1} |e_j|^{-4}\right).$$

The next step gives estimation for $\sum_{j=1}^{\infty} |b_j|^{-3} |e_j|^{-2}$ through $\sum_{j=1}^{\infty} |e_j|^{-3}$. Hence, the series $\sum_{j=1}^{\infty} |b_j|^{-3}$ converges. Therefore, the series $\sum_{k=1}^{\infty} |a_k|^{-3}$ is convergent, since it is majorized by $\sum_{j=1}^{\infty} |b_j|^{-3}$.

In order to prove divergence of $\sum_{k=1}^{\infty} |a_k|^{-2}$ let us choose such s that each cell of the lattice $\mathcal{Q}_s$ contains points of A. It is always possible to do it because the points a_k are uniformly distributed in $\mathbb{C}$. The rest proof is based on divergence of $\sum_{j=1}^{\infty} |e_j|^{-2}$.

The lemma is proved.

Corollary. *It follows from the theory of meromorphic functions and Lemma A.1 that a meromorphic function $f(z)$ having simple poles at a_k with the residuum 1 has the form $f(z) = F_1(z) + G_1(z)$, where $F_1(z)$ has the form (4.4), $G_1(z)$ is an entire function.*

Lemma A.2. *The series*

$$\sigma_2(z) := \sum_{k=1}^{\infty} \left[(z - a_k)^{-2} - a_k^{-2} \right] + z^{-2}$$

converges absolutely and almost uniformly in $\mathbb{C} \backslash A$. The function $\sigma_2(z)$ satisfies the property $\mathcal{P}^0$.

P r o o f. Following Lemma A.1 we introduce the series

$$\sum_{j=1}^{\infty} \left[(z - b_j)^{-2} - b_j^{-2} \right] + z^{-2} \tag{A 1}$$

which majorizes $|\sigma_2(z)|$ in $\mathbb{C} \backslash B$, where $B := \{b_j\}_{j=0}^{\infty}$. Despite of $\mathbb{C} \backslash B \subset \mathbb{C} \backslash A$ it is sufficiently to estimate $|\sigma_2(z)|$ in $\mathbb{C} \backslash B$, since we can change by translation the part of B not belonging to $\mathbb{C} \backslash A$. So let z belongs to a compact subset $K \subset\subset \mathbb{C} \backslash B$. We have

$$\sum_{j=1}^{\infty} \left| (z - b_j)^{-2} - b_j^{-2} \right| \leq \sum_{j=1}^{\infty} \left| (z - b_j)^{-2} - (z - e_j)^{-2} \right| + \tag{A 2}$$

$$\sum_{j=1}^{\infty} \left| (z - e_j)^{-2} - e_j^{-2} \right| + \sum_{j=1}^{\infty} \left| b_j^{-2} - e_j^{-2} \right| .$$

Similar to Lemma A.1 the third series is estimated by the convergent series $\sum_{j=1}^{\infty} |e_j|^{-3}$. It follows from the theory of elliptic functions that the second series of (A 2) converges absolutely and uniformly in K.

Let us study the first series (A 2). Let z belongs to the k-th cell of the lattice $\mathcal{Q}_s$. Estimate

$$\sup_k \sum_{j=1}^{\infty} \left| (z - b_j)^{-2} - (z - e_j)^{-2} \right| =$$

$$\sup_k \left(\left| (z - b_k)^{-2} - (z - e_k)^{-2} \right| + \sum_{j \neq k} \left| (z - b_j)^{-2} - (z - e_j)^{-2} \right| \right) .$$

We have

$$\sum_{j\neq k}\left|(z-b_j)^{-2}-(z-e_j)^{-2}\right|\leq\frac{s}{\sqrt{2}}\sum_{j\neq k}\left(|z-e_j|^{-2}|z-b_j|^{-1}+|z-b_j|^{-2}|z-e_j|^{-1}\right). \tag{A 3}$$

The following equalities

$$|z-b_j|\leq|z-e_j|+\frac{s}{\sqrt{2}} \text{ and } |z-b_j|\geq\left||z-e_j|-\frac{s}{\sqrt{2}}\right|$$

hold. Hence, the series (A 3) is estimated by the following series

$$\sum_{j\neq k}|z-e_j|^{-1}\left||z-e_j|-\frac{s}{\sqrt{2}}\right|^{-2}+\sum_{j\neq k}|z-e_j|^{-2}\left||z-e_j|-\frac{s}{\sqrt{2}}\right|^{-1}.$$

The last series is estimated by the series $\sum_{j\neq k}|z-e_j|^{-3}$ uniformly convergent in K.

The lemma is proved.

Along similar lines we can prove the following

Lemma A.3. *The series*

$$\sigma_n(z):=\sum_{k=0}^{\infty}(z-a_k)^{-n} \;\; (n=3,4,...)$$

converges absolutely and almost uniformly in $\mathbb{C}\backslash A$. *The function* $\sigma_n(z)$ *satisfies the property* $\mathcal{P}^0$.

Appendix B

In the present section we discuss the $\mathbb{R}$-linear problem (3.9) for the infinitely connected domain D. Let us note that the $\mathbb{R}$-linear problem for finitely connected domains was studied by Bojarski (1958), Mikhajlov (1970), Mityushev (1985, 1997d).

Let us introduce the functions

$$\omega(z):=\int_w^z\psi(z)\,dz,\;\omega_k(z):=\int_{a_k+r}^z\psi_k(z)\,dz,\;k=0,1,...\,,$$

where w is a fixed point of the domain D. Integrating (3.9) we arrive at the relation

$$\omega(t)=\omega_k(t)-\nu\overline{\omega_k(t)}+c_k,\;|t-a_k|=r,\;k=0,1,..., \tag{B 1}$$

where $c_k = \int_w^{a_k+r} \psi_k(z)\, dz$ are constants.

Following Bojarski (1958) we introduce the function

$$U(z) := \omega(z), \ z \in D,$$

$$U(z) := \omega_k(z) - \nu\overline{\omega_k(z)} + c_k, \ |z - a_k| \le r, \ k = 0, 1, \ldots .$$

This function satisfies the following partial differential equation

$$U_{\overline{z}} + \mu \overline{U}_z = 0, \tag{B 2}$$

where $\mu = 0$ in D, $\mu = \nu$ in D_k, $k = 0, 1, \ldots$. Inequality $|\mu| < 1$ implies that equation (B 2) is of elliptic type. We have

$$\lim_{z \to t \ z \in D_k} U(z) = \lim_{z \to t \ z \in D} U(z).$$

Therefore, one can consider (B 2) as an elliptic equation in a class of generalized functions on $\mathbb{C}$. We now proceed to investigate $U(z)$ at infinity. It follows from the maximum principle that

$$|U(z)| \le |U(t)| \ \text{ for } \ |z| \le R,$$

where $R > 0$, t is an appropriate point of ∂D depending on R. We have

$$|U(t)| \le (1 + |\nu|)\, |\omega_k(t)| + |c_k|, \ |t - a_k| = r,$$

$$|\omega_k(t)| \le 2Tr, \ |c_k| \le |a_k + r - w|\, M,$$

where $T := \sup_k |\psi_k(t)|$, $M := \sup |\psi(t)|$. Hence $|U(z)| \le c\,|z|$, as $z \to \infty$, where c is a positive constant. The general Liouville theorem implies that $U(z)$ is a $\mathbb{R}$-linear function $\alpha z + \beta \overline{z} + \gamma$. Therefore, $\omega(z)$ is a linear function in D, and $\psi(z) = \omega'(z) = \textit{constant}$ in D. Substituting this constant into (3.9) we obtain that $\psi_k(z) \equiv 0$ $(k = 0, 1, \ldots)$.

References

Berdichevskij, V.L. 1983 Variational principles of continuum mechanics. Moscow: Nauka [in Russian].

Bergman, D.J. 1978 The dielectric constants of a composite material - a problem in classical physics. Phys.Rep. C43, 377-407.

Bergman, D.J. 1982 Rigorous bounds for the complex dielectric constant of a two-component composite. Ann. Phys. 138, 78-114.

Bergman, D.J. 1985 Bulk physical properties of composite media. In: Les Méthodes de l'homogénéisation: theorie et applications en physique, Edition Eyrolles, Paris, 1-128.

Bergman, D.J. & Dunn, K.J. 1992 Bulk effective dielectric constant of a composite with periodic microgeometry. Phys.Rev.B, 45, 13262-13271.

Bojarski, B. 1960 On the generalized Hilbert boundary value preoblem. Soobschenia AN GrSSR, 25, 385-390 [in Russian.]

Clark, K.E. & Milton, G.W. 1995 Optimal bounds correlating electric, magnetic and thermal properties of two-phase, two-dimensional composites. Proc.R.Soc.Lond. A448, 161-190.

Golden, K. & Papanicolaou, G. 1983 Bounds for the effective parameters of heterogeneous media by analytic continuation. Comm.Math.Phys. 90, 473-491.

Hurwitz, A. 1964 Allgemeine funktionentheorie und elliptishe funktionen. Berlin etc.: Springer-Verlag.

Jikov, V.V., Kozlov, S.M. & Oleinik, O.A. 1994 Homogenization of differential operators and integral functionals. Berlin etc.: Springer-Verlag.

Markov, K.Z. & Zvyatkov, K.D. 1991 Optimal third-order bounds on the effective properties of some composite media, and related problems. Advances Mech. 14, N 4, 3-46.

McPhedran, R.C. 1986 Transport properties of cylinder pairs and of the square array of cylinders. Proc.R.Soc.Lond. A408, 31-43.

McPhedran, R.C. & Milton, G.W. 1987 Transport properties touching of cylinder pairs and of the square array of touching cylinders. Proc.R.Soc.Lond. A411, 313-326.

McPhedran, R.C., Poladian, L. & Milton, G.W. 1988 Asymptotic studies of closely spaced, highly conducting cylinders. Proc.R.Soc.Lond. A415, 185-196.

Mikhajlov, L.G. 1970 A new class of singular integral equations. Groningen: Wolters-Noordhoff Publ.

Milton, G.W. 1981 Bounds on the complex permittivity of a two-component composite material. J.Appl.Phys. 52, 5286-5293.

Mityushev, V. 1985 On solution to Markushevich's boundary value problem for a circular domains. Vesti Akad.navuk BSSR. Ser.Phys.-Math., N 1, 119. Preprint N 4073-84, VINITI [in Russian.]

Mityushev, V. 1995a Rayleigh's integral and square array of cylinders. Arch.Mech. 47, 1169-1181.

Mityushev, V.1995b Steady heat conduction of two-dimensional composites. IX Symp. Heat Mass Transfer. Augustow 1995, Poland, 93-100.

Mityushev, V. 1996 Application of functional equations to the effective thermal conductivity of composite materials. Slupsk: WSP [in Polish].

Mityushev, V. 1997a Transport properties of regular arrays of cylinders. ZAMM. 77, 115-120.

Mityushev, V. 1997b A functional equation in a class of analytic functions and composite materials. Demostratio Math. 30, 63-70.

Mityushev, V. 1997c Transport properties of finite and infinite composite materials, and Rayleigh's sum. Arch.Mech. 49, 345-358.

Mityushev, V. 1997d Riemann problem on the double of a multiply connected region. Ann.Pol.Math., 67, 1-14.

Rayleigh, Lord. 1892 On the influence of obstacles arranged in rectangular order upon the properties of medium. Phil.Mag. 34, 481-502.

Sangani, A.S. & Yao, C. 1988 Transport processes in random arrays of cylinders. I. Thermal conduction. Phys.Fluids 31, 2426-2434.

Torquato, S. 1991 Random heterogeneous media: microstructure and improved bounds on effective properties. Appl.Mech.Rev. 44, 37-76.

Fig.1. A composite material with circular inclusions.