Constructive methods]CONSTRUCTIVE METHODS FOR BOUNDARY VALUE PROBLEMS FOR ANALYTIC FUNCTIONS

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1. Introduction

In this paper we describe a number of constructive methods for analytic functions following the ideas of the book [26] and recent new papers. These methods extend the possibility of using mathematical constructions within different directions of mathematics, as well as mechanics, physics, chemistry, biology, economics etc. This field is too wide to be presented in a short paper. Thus we restrict ourselves by boundary value problems for analytic functions and some related problems of porous media mechanics and composite materials.

What is for us the meaning of the expression *explicit* or *closed form solution*? To get the closed form solution one have to construct the formula which contains a finite set of elementary and special functions, arithmetic operations, compositions, integrals, derivatives and even series. Besides all the objects in the formula ought to have precise meaning. At last, domains for parameters, as well as all functions, integrals, derivatives etc. have to be explicitly determined. It shoud be shown also that these domains and (if necessary) their intersections are non-empty.

This approach seems to be slightly nontraditional. For instance, in the classical books a form in series is not supposed to be a closed one. At the same time these books allow to have special functions in the solutions' formulas. But not all such functions has integral representations.

Here we discuss certain nonlinear boundary value problems. We choose them because of two reasons. First of all, they are natural generalizations of corresponding linear problems. Second, they appeared due to special type of applications. It should be noted that the corresponding linear boundary value problems are widely described in the literature. Classical book by Gakhov [9], by Muskhelishvili [29], as well as more recent books by Prössdorf [35], by Mikhlin & Prössdorf [27], by Gokhberg & Krupnik [10], by Meister [21], by Begehr & Wen [4], [40], by Lu [20] and others present different approaches to linear problems and numereous applications of these problems.

The goals of our paper are to give an analysis of the modern state in the considered branch of mathematics, to discover new features appeared at the passage from linear to nonlinear case, to determine new areas for application and really new problems which is necessary to investigate. From this point of view we pay attention of the readers to certain linear problems which are still of great interest.

2. Linear boundary value problems. A continuation

It is commonly known that linear theory of boundary value problems for analytic function is closed to be completed. Anyway, we have to pay attention of readers to a couple of linear problems whose study have to be continued. This interest appeared due to new type of applications appeared recently. These problems are

- boundary value problems for multiply connected domains;
- boundary value problems with infinite index and for infinitely connected domains.

The problem of the first type was discussed even in the book [22]. It was shown the following features of the considered problems. First, the solutions of these problems are in general multi-valued functions. To overcome this difficulty one should choose an appropriate single-valued branch. A deep discussion connected with this fact is presented in [6]. Second, there are only few exact solutions of the boundary value problems for multiply connected domains. We have pointed out the celebrate Villat's formula gave the solution of the Schwarz problem for the concentric annulus (see e.g. [3]) and the solution of Riemann-Hilbert problem for multiply connected circular domains presented in [24]. The main element of the construction for the later problem is the explicit form of Schwarz operator.

Let $\mathbf{D} := \hat{\mathbb{C}} \setminus (\cup_{k=1}^{n} \mathrm{cl}\, \mathbf{D}_k)$ be a circular plane multiply connected domain, $\mathbf{D}_k = z \in \hat{\mathbb{C}} : |z - a_k| < r_k$. Let the formula

$$z^*_{(k_m k_{m-1} \ldots k_1)} := \left(z^*_{(k_{m-1} \ldots k_1)} \right)^*_{(k_m)} \tag{1}$$

defines successive inversions with respect to circles $\mathbf{T}_{k_1} =: \partial \mathbf{D}_{k_1}, \ldots, \mathbf{T}_{k_m} =: \partial \mathbf{D}_{k_m}$ with indeces $k_1, \ldots, k_m$. The transformation (1) is either a Möbius transformation (their collection forms a group $\mathcal{G}'$), or is an anti-Möbius transformation, i.e., is Möbius with respect to $\bar{z}$. The later form a group $\mathcal{F}'$. Let $\mathcal{G}$ and $\mathcal{F}$ denote $\mathcal{G}'$ and $\mathcal{F}'$ without the identity element. The Schwarz operator defines an analytic function $F(z)$ in a multiply-connected circular domain $\mathbf{D}$ via the given values $f(t)$ of its real part, i.e., $\Re\, F(t) = f(t)$, on the boundary $\partial\mathbf{D}$ of the domain. It can be represented in the form (see [24], [26])

$$F(z) = \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbf{T}_k} f(\zeta)\left\{\sum_{\gamma_j\in\mathcal{G}}\left[\frac{1}{\zeta-\gamma_j(w)} - \frac{1}{\zeta-\gamma_j(z)}\right]\right.$$
$$\left. + \left(\frac{r_k}{\zeta-a_k}\right)^2 \sum_{\gamma_j\in\mathcal{F}}\left[\frac{1}{\overline{\zeta-\gamma_j(\bar{z})}} - \frac{1}{\overline{\zeta-\gamma_j(\bar{w})}}\right] - \frac{1}{\zeta-z}\right\} d\zeta \tag{2}$$
$$+\frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbf{T}_k} f(\zeta)\frac{\partial A}{\partial\nu}(\zeta)d\zeta + \sum_{m=1}^{n} A_m\left[\log(z-a_m)+\psi_m(z)\right] + iB,$$

where w is an arbitrary fixed point of the domain, the constants A_m are explicitly determined by an additional system of linear algebraic equations, B is an arbirary real constant. The functions A, and ψ_m are determined by the mappings of the groups $\mathcal{G}'$ or $\mathcal{F}'$ (see [26]).

As for the second class of the problems it is highly related to the boundary value problem with infinite index [11]. The common for these problems is that their solvability are determined by the properties of certain classes entire and meromorphic functions. The base for these problems is presented in the book [11]. The solution of the following $\mathbb{C}$-linear conjugation problem is investigated there: let L be a simple smooth unbounded arc which is "near"-ray at infinity, let $\log G$ and g be given Hölder-continuous functions on L, let the argument of G is locally Hölder-continuous function on L, which behaves at infinity as

$$\arg G(t) \sim \lambda|t|^{\rho},\ \lambda\in\mathbb{R},\ 0<\rho<\infty.$$

The problem is to determine a function Φ, analytic in the complex plane cut along L, continuous up to both sides of L, satisfying on L the following boundary condition:

$$\Phi^{+}(t) = G(t)\Phi^{-}(t) + g(t),\quad t\in L. \tag{3}$$

Explicit formulas for the bounded solutions to (3), containing entire functions, were found. The classes of such functions were described for different classes of solutions. It was shown, in particular, that for $\lambda > 0$ the problem has countable set of linear independent solutions, and for $\lambda < 0$ there exists a unique solution if and only if countable set of solvability conditions does satisfy.

The boundary value problems for infinitely connected domains has the same nature as the above described problem. We discuss these problems in Section 4.

3. Nonlinear boundary value problems

Let us briefly describe the results for nonlinear boundary value problems for analytic functions. Some of these problems were stated in the article [31] (see also [26], [32] and references therein). The most investigated among them is so called *general nonlinear conjugation problem of power type*: given simple closed curve divided a complex plane onto two domains D^+ and $D^- \ni \infty$ and Hölder continuous functions

$G(t) \neq 0, g(t)$, $t \in L$, find two single-valued analytic functions in the corresponding domains, continuous up to their boundaries, i.e., $\Phi^{\pm} \in \mathcal{C}_{\mathcal{A}}(D^{\pm})$, satsfying the following boundary conditions:

$$\left(\Phi^{+}(t)\right)^{\alpha} = G(t)\left(\Phi^{-}(t)\right)^{\beta} + g(t),\ t \in L. \tag{1}$$

It was appeared that the solvability of the problem (1) highly depends on the distribution of zeroes of unknown functions into domains and on their boundaries. Therefore it is customary to introduce the following classes of functions:

$$\mathcal{A}_m(D) := \{\phi \in \mathcal{C}_{\mathcal{A}}(D) : m \text{ is a divisor of multiplicity of any zero of } \phi\}.$$

For functions from $\mathcal{A}_m(D)$ with prescribed number of zeros we use the the notation

$$\mathcal{A}_m^k(D) := \{\phi \in \mathcal{A}_m(D) : n_D(\phi) = m \cdot k,\}$$

where $n_D(\phi)$ denotes the number of zeros of the function ϕ in D. For the pairs of functions in two complimentary domains the notations $\mathcal{A}_{m,n}$, and $\mathcal{A}_{m,n}^{k,l}$ will be used. If the functions are allowed to have boundary zeros, then the corresponding classes will be denoted $\tilde{\mathcal{A}}_{m,n}$, and $\tilde{\mathcal{A}}_{m,n}^{k,l}$.

The strategy of the solution of the problem (1) is to find the suitable classes of functions in which the initial problem is equivalent to certain linear boundary value problem. The most simple case of solvability of (1) appears when the following equation has a solution in nonnegative integers k, l:

$$\chi = \alpha k + \beta l, \tag{2}$$

where $\chi := \mathrm{wind}_L\, G$ is a Cauchy index (winding number) of the coefficient G in (1). By fixing k and l arbitrary points in the corresponding domains being zeros of the solution (for k and l satisfying (2) one can determine then a unique solution of the initial boundary value problem.

If k, l do not satisfy (2) then it is still possible to find solutions of (1), but these function should necessarily have zeros on L. It can also happend that certain solvability conditions should valid (see [26]).

Another fairly investigated type of the nonlinear boundary value problem is *Riemann-Hilbert problem of power type*. It can be formulated as follows. Let $\tau : \mathbb{T} \to L$, $\tau = \tau(s)$, be a given parametrization of certain simple smooth curve L. Given Hölder continuous, 2π-periodic functions $\lambda(s) \neq 0, f(s)$ on the unit circle $\mathbb{T}$, find a single-valued analytic function $\omega \in \mathcal{C}_{\mathcal{A}}(D)$ satisfying the following boundary condition:

$$\Re\left\{\overline{\lambda(s)}\omega^{p}(\tau(s))\right\} = f(s),\ s \in L. \tag{3}$$

The problem (3) can be considered in rather general domains (not only in simply-connected), but for only few types of domains one can find closed form solution. It is connected with the possibility of explicit conformal representation of a domain or with possibility to solve explicitly the Schwarz problem in the corresponding domain (see above discussion concerning construction of Schwarz operator).

The idea of solving the problem (3) is also in finding suitable classes of functions in which this problem appears to be equivalent to a linear boundary value problem. Additional questions to be discussed are study of some "symmetry" properties of the solution, determination of special branches of multi-valued functions, showing that the constructed solutions of initial problem actually belongs to choosen classes.

The results for the problem (3) are given mainly for the case of the unit disc. It is necessary to point out that in the nonlinear case the problems (3) and (1) are in general not equivalent even for L being the unit circle $\mathbb{T}$. In particular, if the exponent p is a purely imaginary number, then the problem (3) can have a solution with countable collection of zeros in the domain.

One more problem took a great attention of specialists due to its application in mechanics (see e.g., [7]). It is a *multiplication type problem.* Let us formulate it in the case of a compound contour. Let L be a piece-wise smooth connected curve consicting of a finite number of open smooth arcs having no common points except possibly their end-points. Without loss of generality one can suppose $\infty \notin L$. An end-point of at least one arc will be called a knot of L. An order of knot is the number of arcs ending at this point. Let us assume additionally that all knots $\alpha_1, \dots, \alpha_\nu$, $\nu \in \mathbb{N}$, of L have even orders. In this case L divides the complex plane $\mathbb{C}$ onto the finite number of simply connected domains $D_1, \dots, D_\mu$. Besides L is so called Euler graph and $\mathbb{C} \setminus L$ is a 2-colourable chart, i.e. it can be coloured by using only two colours in such a way that any neighbouring domains have different colour. Denote the union of the domains coloured by one of the colours by D^+, and $\mathbb{C} \setminus D^+$ by D^-. The orientation on L is induced by D^+. Consider the following classes of functions: $\mathcal{B}$ set of all pairs of functions, analytic and bounded in $\mathbb{C} \setminus L$, continuous up to $L \setminus \{\alpha_1, \dots, \alpha_\nu\}$, nonvanishing on $L \setminus \{\alpha_1, \dots, \alpha_\nu\}$; $\mathcal{B}^{k,l} \subset \mathcal{B}$ is a subclass of $\mathcal{B}$ of those functions having exactly k zeros in D^+ and l zeros in D^-.

The *multiplication type problem* consists in the following: for a given countour L and a Hölder-continuous function f on L find a pair of functions $(\Phi^+, \Phi^-) \in \mathcal{B}$ or $\mathcal{B}^{k,l}$ satisfying the following boundary condition

$$\Phi^+(t) \cdot \Phi^-(t) = f(t),\ t \in L \setminus \{\alpha_1, \dots, \alpha_\nu\}. \tag{4}$$

The most crucial point is to solve the simple problem of the type (4) (see e.g. [16], [26])

$$\Phi^+(t) \cdot \Phi^-(t) = 1,\ t \in L \setminus \{\alpha_1, \dots, \alpha_\nu\}. \tag{5}$$

The solution of the problem (5) can be presented in the form

$$\begin{cases} \Phi^+(z) = \prod\limits_{j=1}^{k} \left(1 - \frac{z}{z_j^+}\right) \prod\limits_{s \in I_0} (z-\alpha_s)^{m_s} \exp\left(\sum\limits_{s \in I_0} F_s(\frac{1}{z-\alpha_s}) \right), \\ \Phi^-(z) = z^{-l_0} \prod\limits_{j=1}^{l_1} \left(1 - \frac{z_j^-}{z}\right) \prod\limits_{s \in I_0} (z-\alpha_s)^{-m_s} \exp\left(- \sum\limits_{s \in I_0} F_s(\frac{1}{z-\alpha_s}) \right), \end{cases} \tag{6}$$

where $l_0 + l_1 = l$, a function $F_s(\frac{1}{z-\alpha_s})$ is an entire function with respect to the knot

α_s, and the set of indeces I_0 is choosen due to geometrical properties of the curve L near the point α_s.

We have to mention also the *modulus problem*: given a nonegative Hölder-continuous function f on a closed smooth curve L find a function $\omega \in \mathcal{C}_{\mathcal{A}}(D)$ satisfying the following boundary condition:

$$|\omega(t)| = f(t),\ t \in L.$$

The modulus problem is highly related to the problem of conformal representation of a domain (see e.g. [18], [33], [34]). Both problems can be reduced to a nonlinear integral equations investigated numerically. We have to mention here the recent article [19]. The only few types of such integral equations can be studied constructively. Therefore this is the problem for further considerations.

Among the problems which for the moment have no complete solution we have to mention *linear fractional boundary value problem*

$$\Phi^+(t) + a(t)\Phi^-(t) + b(t)\Phi^+(t) \cdot \Phi^-(t) = c(t),\ t \in L. \tag{7}$$

The problem (7) appeared to be equivalent to another open problems of analysis, e.g. *generalized modulus problem*: given Hölder-continuous functions $f(t) \geq 0, \alpha(t)$ on a closed smooth curve L find a function $\omega \in \mathcal{C}_{\mathcal{A}}(D)$ satisfying the following boundary condition:

$$|\omega(t) - \alpha(t)| = f(t),\ t \in L;$$

to *linear conjugation four-elements vector-matrix boundary value*; to $\mathbb{R}$-*linear conjugation problem*; to *the problem of vector-matrix factorization.*

At last the constructive results for linear and nonlinear boundary value problems are used as a part of compound investigation, in particular in different type applied problems (see e.g. [8], [36], [38], [39]).

4. Some applications to porous media and composite materials

Some problems of the porous media mechanics and composite materials can be stated as two-dimensional problems for Laplace's equations and others [1]. Using complex potentials one can arrive at the $\mathbb{R}$-linear problem [26]

$$\phi^+(t) = \phi^-(t) - \rho_k\overline{\phi^-(t)} + g(t),\ t \in \partial D_k, k = 1, 2, ..., \tag{1}$$

where $D^- = \cup_k D_k$, D_k are simply connected domains (inclusions), $D^+ = \hat{\mathbb{C}} \setminus (\mathrm{cl}D^-)$. The constants ρ_k are called the contrast parameters; they describe physical properties of the inclusions. The known function $g(t)$ models sources and sinks.

The problem (1) for finitely connected circular domain D^+ has been solved in [26] in terms of the Poincare series. In order to determine the effective properties of the medium we have to solve (1) for infinitely connected domains. As it was shown in [14] a composite material is homogenized only for periodical or statistically periodical structures. In the first case we arrive at the problem (1) for a finitely connected domain on torus. Constructive algorithm to solve (1) for circular domains on torus has been proposed in [23]. Problem (1) for identical disks arbitrary distributed on the plane and for equal $\rho_k = \rho$ has been solved in [25] up to $O(\rho^3)$, as ρ tends to zero. Statistical perturbations of the periodical structures have been studied in [5] by using of the "shaking model". Let us note that in general statistical boundary value problems are not constructively investigated. It is important in applications to study (1), for instance in the case, when all inclusions have the same shape and uniformally distributed on the plane. The inclusions may have different sizes and orientation which can be also determined by distributions.

The next interesting problem is the problem of the inclusions with fractal boundary and fractal character of a set of inclusions. Special cases, when infinite number of circular inclusions is located in a bounded domain have been investigated in [2].

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