Transport properties of doubly periodic arrays of circular cylinders and optimal design problems

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Abstract

We solve an $\mathbb{R}$-linear problem for a multiply connected circular domain in a class of doubly periodic functions in analytic form by a method of functional equations. This problem models transport properties of two-dimensional composite materials made from a collection of disks imbedded in an otherwise uniform host.

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1 Introduction

Transport properties of two-dimensional doubly periodic composite materials made from a collection of non-overlapping circular disks imbedded in an otherwise uniform host are considered. There have been many theoretical approaches to this problem. One of them is to consider specific periodic structure, and to solve approximately or analytically the transport problem. Grigiluk & Filshtinskij [4] applied a method of singular integral equations to boundary value problems for doubly periodic functions. McPhedran et al. [9] (see also papers cited therein) studied the square and triangle arrays of disks. Having based on the classical paper of Lord Rayleigh [14] they reduced the problem to an infinite set of linear algebraic equations. Similar results are also represented in [4] and works cited therein. Kolodziej [6] applied the method of collocations to study a wide class of doubly periodic composite materials. Sangani & Yao [16] developed the method of singular solutions and reduced the problem to an infinite set of equations. In the works [4, 9, 16, 14] and others the infinite sets had been truncated to give various low-ordered formulae for the effective transport properties.

A method of functional equations has been applied in [10, 11, 12]. The effective conductivity tensor has been written in analytic form for an arbitrary doubly periodic array of disks. In the present paper we proceed to use the method of functional equations. This method is applied to a doubly periodic array with each unit cell containing N circular inclusions whose size, location and the properties are completely arbitrary. The crucial point is based on solution to an $\mathbb{R}$-linear problem. The effective conductivity tensor is written in analytic form. The method of functional equation is closely related to a method of perturbation. See Sec.3. One can find advanced applications of the perturbation theory in [17].

A typical problem of shape optimization is to minimize an energy function over the set of the characteristic inclusion functions which take unity in their corresponding domain and zero elswhere. Let us note that shape and number of inclusions are not fixed in this statement. Such optimal design problems are successfully solved by homogenezation methods (see [1, 7, 2] and 'references cited therein.) In Sec.6 we discuss another class of optimal design problems, when it is necessary to locate N circular disks with given sizes and properties in the unit cell representing a composite material. Our goal is to determine such a location of the inclusions that anisotropy of the homogenized material reaches the maximum value.

In Sec.6 we determine the principal axes and the angle between one of them and the x axis. We introduce the anisotropy coefficient involving only geometrical parameters. As to our knowing such quantitative values were not used earlier. This study is useful in technical applications, because the major and minor axes are the most effective directions of conductivity and isolation, respectively. So using our formulae a designer can project complex fibre composite materials to reach optimal properties in given directions.

2 Formulation of the boundary value problem

We consider a lattice $\mathcal{Q}$ which is defined by the two-fundamental translation vectors α and $i\alpha^{-1}$ $(\alpha > 0,\ i^2 = -1)$ in the complex plane $\mathbb{C} \cong \mathbb{R}^2$. The zero-th cell $Q_{(0,0)}$, the basis of $\mathcal{Q}$, is the rectangle $\{z = t_1\alpha + t_2 i\alpha^{-1} \in \mathbb{C} :\ -1/2 < t_p < 1/2,\ p = 1, 2\}$, where $z = x + iy$ is a complex variable. For the area holds $\left|Q_{(0,0)}\right| = 1$. Let $\mathcal{E} := \cup_{\mathbf{j}} \{e_{\mathbf{j}}\}$ be a doubly ordered set of the numbers $e_{\mathbf{j}} := (m_1\alpha + m_2 i\alpha^{-1})$, where $\mathbf{j} = (m_1, m_2)$, m_1 and m_2 are integer, $e_{(0,0)} = 0$. The lattice $\mathcal{Q}$ consists of the cells $Q_{\mathbf{j}} = Q_{(0,0)} + e_{\mathbf{j}} := \left\{ z \in \mathbb{C} : \ z - e_{\mathbf{j}} \in Q_{(0,0)} \right\}$.

Let us consider mutually disjoint disks $\mathbb{B}_k := \{ z \in \mathbb{C} : \ |z - a_k| < r_k \}$ $(k = 1, 2, ..., N)$ in the zero-th cell $Q_{(0,0)}$. Let $D := Q_{(0,0)} \backslash \left(\cup_{k=1}^{N} \mathbb{B}_k \cup \mathbb{T}_k \right)$, where $\mathbb{T}_k := \{ t \in \mathbb{C} : \ |t - a_k| = r_k \}$. Here and after we use the letter z for a variable in a domain, t - on the boundary of a domain. We study the conductivity of the doubly periodic composite material, when the domains $D + e_{\mathbf{j}}$ and $\mathbb{B}_k + e_{\mathbf{j}}$ are occupied by materials of conductivities $\lambda = 1$ and $\lambda_k > 0$, respectively. We find the potentials $u(z)$ and $u_k(z)$ to be harmonic in $D + e_{\mathbf{j}}$ and $\mathbb{B}_k + e_{\mathbf{j}}$ $(k = 1, 2, ..., N; \ e_{\mathbf{j}} \in \mathcal{E})$ with the conjugation conditions:

$$u = u_k, \ \frac{\partial u}{\partial n} = \lambda_k \frac{\partial u_k}{\partial n} \text{ on } |t - a_k| = r_k, \ k = 1, 2, ..., N, \tag{1}$$

where $\frac{\partial}{\partial n}$ is a normal derivative. The function $u(z)$ is quasiperiodic in $\mathbb{C}$:

$$u(z + \alpha) = u(z) + \alpha, \ u\left(z + i\alpha^{-1}\right) = u(z). \tag{2}$$

The equalities (2) mean that we fix the x-direction of the external current.

The problem (1), (2) is equivalent to the following $\mathbb{R}$-linear problem [13]

$$\varphi(t) = \varphi_k(t) - \rho_k \overline{\varphi_k(t)} - t, \ |t - a_k| = r_k, \ k = 1, 2, ..., N, \tag{3}$$

where $\rho_k := (\lambda_k - 1) / (\lambda_k + 1)$, the unknown functions $\varphi(z)$ and $\varphi_k(z)$ are analytic in D and $\mathbb{B}_k$, respectively, and continuously differentiable in the closures of these domains. The function $\varphi(z)$ is quasiperiodic in $\mathbb{C}$:

$$\varphi(z + \alpha) - i\gamma_1 = \varphi(z) = \varphi\left(z + i\alpha^{-1}\right) - i\gamma_2,$$

where γ_1 and γ_2 are real constants. The harmonic and analytic functions are related by the identities

$$u(z) = Re\ (\varphi(z) + z), \ u_k(z) = \frac{2}{1 + \lambda_k} Re\ \varphi_k(z). \tag{4}$$

We have to prove only that $\varphi(z)$ is single-valued in the multiply connected domain D. Using (3) we have

$$\int_{\mathbb{T}_k} \varphi(t)\, dt = \int_{\mathbb{T}_k} \varphi_k(t)\, dt - \rho_k \int_{\mathbb{T}_k} \overline{\varphi_k(t_k^*)} dt - \int_{\mathbb{T}_k} t dt = 0,$$

where $t_k^* := r_k^2 / (t - a_k) + \overline{a_k}$ is the inversion with respect to $\mathbb{T}_k$. Let us note that the functions $\varphi_k(t)$ and $\overline{\varphi_k(t_k^*)}$ are analytically continued into

$|z-a_k| < r_k$ and $|z-a_k| > r_k$, respectively. In order to determine the current $\nabla u(x,y)$, $\nabla u_k(x,y)$ we need only the derivatives

$$\psi(z) := \varphi'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y},\ \psi_k(z) := \varphi_k'(z) = \frac{\partial u_k}{\partial x} - i\frac{\partial u_k}{\partial y}.$$

Differentiating (3) we arrive at the following $\mathbb{R}$-linear problem

$$\psi(t) = \psi_k(t) + \left(\frac{r_k}{t-a_k}\right)^2 \rho_k\overline{\psi_k(t)} - 1,\ |t-a_k| = r_k,\ k = 1,2,...,N, \quad (5)$$

where the function $\psi(z)$ is doubly periodic.

3 The perturbation method

The perturbation method with respect to the parameters ρ_k $(k = 1,2,...,N)$ consists in finding a solution of the problem (3) in the form of the following expansions

$$\varphi(z) = \varphi^0(z) + \sum_{m=1}^{N} \rho_m\varphi_m^1(z) + ...,\ \ \varphi_k(z) = \varphi_k^0(z) + \sum_{m=1}^{N} \rho_m\varphi_{mk}^1(z) + ...$$

By substituting these expansions into (3) and collecting terms with respect to $\rho_{\mathbf{j}}\rho_l...\rho_m$ we obtain a cascade of the jump problems. The zero-th one is

$$\varphi^0(t) = \varphi_k^0(t) - t,\ |t-a_k| = r_k,\ k = 1,2,...,N,$$

where $\varphi^0(z)$ is quasiperiodic. It is easily seen that

$$\varphi^0(z) = 0,\ \varphi_k^0(z) = z$$

is a unique solution to the zero-th problem up to an arbitrary additive constant. The first-order problems are

$$\begin{aligned} \varphi_m^1(t) &= \varphi_{mk}^1(t),\ |t-a_k| = r_k,\ \text{for } k = 1,2,...,N;\ k \neq m, \quad (6)\\ \varphi_m^1(t) &= \varphi_{mm}^1(t) - \left(\frac{r_m^2}{t-a_m} + a_m\right),\ |t-a_m| = r_m, \end{aligned}$$

$m = 1,2,...,N$. The first equality (6) means that the function $\varphi_m^1(z)$ is analytically continued into $|z-a_k| < r_k$ for $k \neq m$. The second relation (6) implies that $\varphi_m^1(z)$ is analytically continued into $0 < |z-a_m| < r_m$ and has a pole at $z = a_m$ of first order. It follows from the theory of elliptic functions [3] and [10] that

$$\varphi_m^1(z) = \varphi_{mk}^1(z) = r_m^2 \left[\alpha^{-1}\eta_1 z - \zeta(z - a_m)\right], \text{ for } k \neq m,$$

where $\eta_1 = 2\zeta(\alpha/2)$, $\zeta(z)$ is a Weierstrass' function. Hence,

$$\varphi_{mm}^1(z) = r_m^2 \left[\alpha^{-1}\eta_1 z - \zeta(z - a_m) + (z - a_m)^{-1} + a_m\right],$$

$$\varphi_k(z) = z + \sum_{m\neq k}^{N} \rho_m r_m^2 \left[\alpha^{-1}\eta_1 z - \zeta(z - a_m)\right] + \tag{7}$$

$$\rho_k r_k^2 [\alpha^{-1}\eta_1 z - \zeta(z - a_k) + (z - a_k)^{-1} + a_k] + o(\rho), \text{ as } \rho \to 0,$$

$$\psi_k(z) = 1 + \textstyle\sum_{m\neq k}^{N} \rho_m r_m^2 [\alpha^{-1}\eta_1 + \mathcal{P}(z - a_m)] +$$

$$\rho_k r_k^2 \left[\alpha^{-1}\eta_1 + \mathcal{P}(z - a_k) - (z - a_k)^{-2}\right] + o(\rho), \text{ as } \rho \to 0.$$

Here $\rho := \max_k |\rho_k|$, $\mathcal{P}(z) = -\zeta'(z)$ is the next Weierstrass' function, m runs from 1 to N, except k, in the sum $\sum_{m\neq k}^{N}$. The second and third formulae (7) represent the complex potential and the current in $|z - a_k| < r_k$ up to $o(\rho)$.

4 The method of functional equations

In Sec.3 a perturbation method has been applied to the $\mathbb{R}$-linear problem (3). The zero-th and first approximations have been calculated. It is possible to continue this procedure and to find higher terms. Actually we shall do it by means of functional equations. This method leads to exact solution of the problem.

We are going to use the conditionally convergent sum $S_2 := \sum_{\mathbf{j}}{}' e_{\mathbf{j}}^{-2}$ and absolutely convergent sums $S_l := \sum_{\mathbf{j}}{}' e_{\mathbf{j}}^{-l}$ $(l = 3, 4, ...)$ corresponding to the lattice $\mathcal{Q}$. The sum $\sum_{\mathbf{j}}{}'$ contains all cell centres $e_{\mathbf{j}}$, apart from the origin, when $\mathbf{j} = (0, 0)$. If l is odd then $S_l = 0$ and S_{2l} are real numbers [14]. We shall use Einstein's summation [18], then

$$S_2 := \alpha^{-1}\eta_1 = \alpha^{-1} 2\zeta(\alpha/2). \tag{8}$$

A rigorous explanation of the definition (8) will be given later. Note that if $\alpha = 1$ then $S_2 = \pi$ [14]. We are going to use also the Eisenstein functions [18]:

$$E_l(z) := \sum_{\mathbf{j}} (z - e_{\mathbf{j}})^{-l}, \ l = 3, 4, ..., \tag{9}$$

where the series (9) converges absolutely and uniformly in each compact subset of $\mathbb{C}\backslash\mathcal{E}$,

$$E_2(z) := \mathcal{P}(z) + S_2. \tag{10}$$

Let us consider the Banach spaces $\mathcal{C}_k$ consisting of the functions continuous on $|t-a_k| = r_k$ with the norm $\|\psi_k\| := \max_{\mathbb{T}_k} |\psi_k(t)|, \ k = 1, 2, ..., N$. And let us consider the closed subspaces $\mathcal{C}_k^+ \subset \mathcal{C}_k$ for which the functions ψ_k have analytic continuation to $|z-a_k| < r_k$. We also introduce the Banach spaces $\mathcal{C}^+$ consisting of the functions $\psi(t) := \psi_k(t) \in \mathcal{C}_k^+$ for all $k = 1, 2, ..., N$ with the norm $\|\psi\| := \max_k \|\psi_k\|$. We shall use the following

Theorem 4.1. *[11]Introduce the operators*

$$\mathbf{T}_{\mathbf{j}k}\psi_k(z) := \left(\frac{r_k}{z-a_k-e_{\mathbf{j}}}\right)^2 \left[\overline{\psi_k\left(\frac{r_k^2}{\overline{\overline{z-a_k-e_{\mathbf{j}}}}} + a_k\right)} - \overline{\psi_k(a_k)}\right], \tag{11}$$

$\psi_k \in \mathcal{C}_k^+, \ k = 1, 2, ..., N, \ \mathbf{j} = (m_1, m_2)$.

i) The series $\sum_{\mathbf{j}} {}' \mathbf{T}_{\mathbf{j}k}\psi_k(z)$ *converges absolutely and uniformly in* $Q_{(0,0)} \cup \partial Q_{(0,0)}$ *for each* $\psi_k \in \mathcal{C}_k^+$.

ii) The function $\sum_{\mathbf{j}} \mathbf{T}_{\mathbf{j}k}\psi_k(z)$ *is analytic in* D*, continuous in* $D \cup \partial D$ *and doubly periodic in* $\mathbb{C}$.

iii) The linear operator $\sum_{\mathbf{j}} {}' \mathbf{T}_{\mathbf{j}k}\psi_k(z)$ *is compact in* $\mathcal{C}_k^+$.

Using this theorem we introduce the function

$$\Phi(z) := \begin{cases} \psi_m(z) - \sum_{k=1}^N \rho_k \sum_{\mathbf{j}}^* \mathbf{T}_{\mathbf{j}k}\psi_k(z) + \rho_m \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m(a_m)} - 1, & |z-a_m| < r_m, \\ & m = 1, 2, ..., N, \\ \psi(z) - \sum_{k=1}^N \rho_k \sum_{\mathbf{j}} \mathbf{T}_{\mathbf{j}k}\psi_k(z), \ z \in D, & \end{cases}$$

where the term with $k = m, \ \mathbf{j} = (0,0)$ is missed in the sum $\sum_{k=1}^N \rho_k \sum_{\mathbf{j}}^*$. It follows from Theorem 4.1 $\Phi(z)$ is analytic in D, all $\mathbb{B}_k$ and doubly periodic in $\mathbb{C}$. Let us calculate the jump

$$\begin{aligned} \Delta &: = \lim_{t\to z \ z\in D} \Phi(z) - \lim_{t\to z \ z\in \mathbb{B}_k} \Phi(z) = \psi(t) - \\ &\left(\frac{r_m}{t-a_m}\right)^2 \rho_m \left[\overline{\psi_m(t)} - \overline{\psi_m(a_m)}\right] - \rho_m \left(\frac{r_m}{t-a_m}\right)^2 \overline{\psi_m(a_m)} + 1, \\ |t-a_m| &= r_m. \end{aligned}$$

Taking into account (5) we obtain $\Delta = 0$. Using Principle of analytic continuation and the general Liouville theorem for doubly periodic functions [3] we conclude that

$$\Phi(z) = \sum_{m=1}^N \rho_m r_m^2 \overline{\psi_m(a_m)} E_2(z-a_m) + C,$$

where C is a constant. In order to be consistent with (7) we put $C = 0$. Here the relation (10) is used. From the definition of $\Phi(z)$ for $|z - a_m| \le r_m$ we have the following set of functional equations

$$\psi_m(z) = \tag{12}$$

$$\sum_{k=1}^{N} \rho_k \left\{ \sum_{\mathbf{j}}^{*} \mathbf{T}_{\mathbf{j}k} \psi_k(z) + r_k^2 \overline{\psi_k(a_k)} \left[E_2(z - a_k) - \delta_{km}(z - a_k)^{-2} \right] \right\} + 1,$$

$$|z - a_m| \le r_m, \; m = 1, 2, ..., N,$$

with respect to $\psi_m \in \mathcal{C}_m^+$. Here δ_{km} is the Kronecker symbol. Let us rewrite (12) as an equation in the space $\mathcal{C}^+$

$$\psi = \mathbf{A}\psi + 1, \tag{13}$$

where $\mathbf{A}\psi(z) := \sum_{k=1}^{N} \rho_k \left\{ \sum_{\mathbf{j}}^{*} \mathbf{T}_{\mathbf{j}k} \psi_k(z) + r_k^2 \overline{\psi_k(a_k)} \left[E_2(z - a_k) - \delta_{km}(z - a_k)^{-2} \right] \right\}$, $\psi(z) := \psi_m(z)$ for $|z - a_m| \le r_m$. It follows from Theorem 1 that $\mathbf{A}$ is a linear compact operator in $\mathcal{C}^+$.

Theorem 4.2. *Equation (13) has the unique solution $\psi = \sum_{n=0}^{\infty} \mathbf{A}^n 1$. The last series converges in $\mathcal{C}^+$.*

Proof It is sufficient to show that $r(\mathbf{A}) < 1$, where $r(\mathbf{A})$ is the spectral radius of the operator $\mathbf{A}$. The operator $\mathbf{A}$ as a compact operator in the Banach space $\mathcal{C}^+$ has the spectrum consisting only of eigenvalues. The inequality $r(\mathbf{A}) < 1$ is valid if and only if there exist a complex number ν such that $|\nu| \le 1$ and the homogeneous equation $\psi = \nu \mathbf{A}\psi$ has a nontrivial solution. This equation can be written in the form

$$\psi_m(z) = \nu \sum_{k=1}^{N} \rho_k \left\{ \sum_{\mathbf{j}} {}^{*}\mathbf{T}_{\mathbf{j}k} \psi_k(z) + r_k^2 \overline{\psi_k(a_k)} \left[E_2(z - a_k) - \frac{\delta_{km}}{(z - a_k)^2} \right] \right\},$$

If $\psi_m(z)$ is a solution of (14) then the function

$$|z - a_m| \le r_m, \; m = 1, 2, ..., N. \tag{14}$$

$$\Psi(z) = \nu \sum_{k=1}^{N} \rho_k \left\{ \sum_{\mathbf{j}} \mathbf{T}_{\mathbf{j}k} \psi_k(z) + r_k^2 \overline{\psi_k(a_k)} E_2(z - a_k) \right\} \tag{15}$$

is doubly periodic and analytic in $D \cup \partial D$. It is easily seen that $\Psi(z)$ and $\psi_k(z)$ satisfy the $\mathbb{R}$-linear problem

$$\Psi(t) = \psi_m(t) + \nu \left(\frac{r_m}{t - a_m} \right)^2 \rho_m \overline{\psi_m(t)}, \; |t - a_m| = r_m, \; m = 1, 2, ..., N. \tag{16}$$

It was shown in [11] that the $\mathbb{R}$-linear problem (16) for $N = 1$ has only zero solution. The same arguments [11] for the general case $N \geq 1$ yield the same result. i.e. $\psi_m(z) \equiv 0, \ m = 1, 2, ..., N$.

The theorem is proved.

Let us introduce the operator

$$\mathbf{W}_{\mathbf{j}k}\psi_k(z) := \left(\frac{r_k}{z - a_k - e_{\mathbf{j}}}\right)^2 \overline{\psi_k\left(\frac{r_k^2}{\overline{z - a_k - e_{\mathbf{j}}}} + a_k\right)},$$

and the series

$$\sum_{\mathbf{j}} \mathbf{W}_{\mathbf{j}k}\psi_k(z) := \sum_{\mathbf{j}} \mathbf{T}_{\mathbf{j}k}\psi_k(z) + r_k^2 \overline{\psi_k(a_k)} E_2(z - a_k) \tag{17}$$

for $\psi_k \in \mathcal{C}_k^+, \ k = 1, 2, ..., N$. Then (12) becomes

$$\psi_m(z) = \sum_{k=1}^{N} \rho_k \sum_{\mathbf{j}} {}' \mathbf{W}_{\mathbf{j}k}\psi_k(z) + 1, \ |z - a_m| \leq r_m, \ m = 1, 2, ..., N. \tag{18}$$

It follows from Theorem 2 that we can apply the method of successive approximations to (12) or (18), which gives the following exact formula

$$\psi_m(z) = 1 + \sum_{k_1=1}^{N} \rho_{k_1} \sum_{\mathbf{j}_1} {}' \mathbf{W}_{\mathbf{j}_1 k_1} 1(z) + \tag{19}$$

$$\sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \rho_{k_1}\rho_{k_2} \sum_{\mathbf{j}_1} {}' \mathbf{W}_{\mathbf{j}_1 k_1} \sum_{\mathbf{j}_2} {}' \mathbf{W}_{\mathbf{j}_2 k_2} 1(z) +, \ |z - a_m| \leq r_m, \ m = 1, 2, ..., N.$$

The series (19) corresponds to the method of perturbations derived in Sec.3. It is worth to note that the compositions of the operators $\mathbf{W}_{\mathbf{j}k}$ are simple in calculations, since they do not contain integral terms. The series (19) involves infinite sums of the elliptic functions. For instance, the function $\psi_m(z)$ up to $O(\rho^3)$ has the form

$$\psi_m(z) = 1 + \sum_{k_1=1}^{N} \rho_{k_1} r_{k_1}^2 E_2(z - a_{k_1}) - \rho_m r_m^2 (z - a_m)^{-2} \tag{20}$$

$$+ \sum_{k_1=1}^{N} \sum_{k_2 \neq k_1}^{N} \rho_{k_1} r_{k_1}^2 \rho_{k_2} r_{k_2}^2 \sum_{\mathbf{j}} {}^* (z - a_{k_1} - e_{\mathbf{j}})^{-2} \overline{E_2\left(\frac{r_{k_1}^2}{\overline{z - a_{k_1} - e_{\mathbf{j}}}} + a_{k_1} - a_{k_2}\right)}$$

$$+ \sum_{k_1=1}^{N} \rho_{k_1}^2 r_{k_1}^4 \sum_{\mathbf{j}} {}^* (z - a_{k_1} - e_{\mathbf{j}})^{-2} \overline{\sigma_2\left(\frac{r_{k_1}^2}{\overline{z - a_{k_1} - e_{\mathbf{j}}}}\right)} + O(\rho^3),$$

as $\rho \rightarrow 0$,

where the term $k_1 = m,\ \mathbf{j} = (0,0)$ is missed in $\sum_{k_1=1}^{N} \sum_{\mathbf{j}}^{*}$. The function [18]

$$\sigma_2(z) := E_2(z) - z^{-2} = \sum_{n=1}^{\infty} (2n-1)\, S_{2n} z^{2(n-1)}$$

is analytic in $Q_{(0,0)}$. Moreover, in accordance with the definition (17) we assume that

$$\sum_{\mathbf{j}} (z - e_{\mathbf{j}})^{-2}\, \overline{\Psi\left(\frac{r_k^2}{\overline{z - e_{\mathbf{j}}}}\right)} := \sum_{\mathbf{j}} (z - e_{\mathbf{j}})^{-2} \left[\overline{\Psi\left(\frac{r_k^2}{\overline{z - e_{\mathbf{j}}}}\right)} - \overline{\Psi(0)}\right] + \overline{\Psi(0)} E_2(z)$$

in (20). Comparing (20) and (7) and using (10) we conclude that the value S_2 is correctly defined by (8).

Another method to solve the set of functional equations (18) is to use the expansions on $r_{k_1}^2 r_{k_2}^2 ... r_{k_n}^2$. We find $\psi_m(z)$ as the following expansion

$$\psi_m(z) = \psi_m^0(z) + \sum_{k_1=1}^{N} r_{k_1}^2 \psi_{k_1 m}(z) + \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} r_{k_1}^2 r_{k_2}^2 \psi_{k_1 k_2 m}(z) + ...$$

Let us represent the operator $\sum_{\mathbf{j}} \mathbf{W}_{\mathbf{j}k} \psi_k(z)$ in the form

$$\sum_{\mathbf{j}} \mathbf{W}_{\mathbf{j}k} \psi_k(z) = \sum_{\mathbf{j}} \overline{\alpha_{lk}} r_k^{2(l+1)} E_{l+2}(z - a_k),$$

where $\psi_k(z) = \sum_{l=0}^{\infty} \alpha_{lk} (z - a_k)^l$ is the Taylor expansion of the function $\psi_k(z)$. Substituting these expansions into (18) we obtain

$$\psi_m^0(z) + \sum_{k_1=1}^{N} r_{k_1}^2 \psi_{k_1 m}(z) + \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} r_{k_1}^2 r_{k_2}^2 \psi_{k_1 k_2 m}(z) + ... =$$

$$\sum_{k \neq m}^{N} \rho_k \sum_{l=0}^{\infty} \overline{\alpha_{lk}} r_k^{2(l+1)} E_{l+2}(z - a_k) \tag{21}$$

$$+\rho_m \sum_{l=0}^{\infty} \overline{\alpha_{lm}} r_m^{2(l+1)} \left[E_{l+2}(z - a_m) - (z - a_m)^{-2}\right] + 1,$$

$$|z - a_m| \ \leq \ r_m,\ m = 1, 2, ..., N,$$

where

$$\alpha_{lk} = \alpha_{lk}^0 + \sum_{k_1=1}^{N} r_{k_1}^2 \alpha_{k_1 m} + \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} r_{k_1}^2 r_{k_2}^2 \alpha_{k_1 k_2 m} + ... \ .$$

Collecting terms with respect to $r_{k_1}^2 r_{k_2}^2 ... r_{k_n}^2$ we arrive at the following recurrent formulae

$$\psi_m^0(z) = 1,$$

$$\psi_{k_1 m}(z) = \rho_{k_1} E_2(z - a_{k_1}) \text{ for } k_1 \neq m,$$

$$\psi_{mm}(z) = \rho_m \left[E_2(z - a_m) - (z - a_m)^{-2} \right],$$
$$...$$

5 Effective conductivity

Let us find the effective properties tensor

$$\Lambda_e = \begin{pmatrix} \lambda_e^x & \lambda_e^{xy} \\ \lambda_e^{xy} & \lambda_e^y \end{pmatrix}$$

of the composite material represented by the zero cell $Q_{(0,0)}$. We consider λ_e^x and λ_e^{xy}. The coefficient λ_e^y will be considered below. Following [10] we have

$$\lambda_e^x = J^x + \sum_{m=1}^{N} \lambda_m J_m^x, \ \lambda_e^{xy} = J^y + \sum_{m=1}^{N} \lambda_m J_m^y$$

where

$$J^x = \int\int_D \frac{\partial u}{\partial x} dxdy, \ J_m^x = \int\int_{\mathbb{B}_m} \frac{\partial u_m}{\partial x} dxdy, \ J^y = \int\int_D \frac{\partial u}{\partial y} dxdy, \ J_m^y = \int\int_{\mathbb{B}_m} \frac{\partial u_m}{\partial y} dxdy.$$

The functions u and u_k satisfy the problem (1), (2). Using the complex potentials we obtain

$$J_m^x = \frac{2}{1+\lambda_m} \int\int_{\mathbb{B}_m} \left[Re\ \varphi_m(z) \right]_x dxdy \text{ and } J_m^y = \frac{2}{1+\lambda_m} \int\int_{\mathbb{B}_m} \left[Re\ \varphi_m(z) \right]_y dxdy.$$

Since $\psi_m(z) = \varphi_m'(z) = \left[Re\ \varphi_m(z) \right]_x - i \left[Re\ \varphi_m(z) \right]_y$, hence we have

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2 \sum_{m=1}^{N} \rho_m \int\int_{\mathbb{B}_m} \psi_m(z)\, dxdy.$$

By virture of the mean value theorem of harmonic functions we obtain

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2 \sum_{m=1}^{N} \rho_m \upsilon_m \psi_m(a_m), \tag{22}$$

where $v_m = \pi r_m^2$ is the area fraction of the inclusions of conductivity λ_m. Basing on (19) we have the exact formula (22) for $\lambda_e^x - i\lambda_e^{xy}$. In particular, using the approximate formula (20) we have

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\sum_{m=1}^N \rho_m v_m + \frac{S_2}{\pi}\sum_{m,k=1}^N \rho_m v_m \rho_k v_k + \frac{1}{\pi}\sum_{m\neq k}^N \rho_m v_m \rho_k v_k \mathcal{P}(a_k - a_m) \tag{23}$$

$$+ \quad \frac{2}{\pi^2}\sum_{m=1}^N \sum_{k_1=1}^N \sum_{k_2\neq k_1}^N \rho_m v_m \rho_{k_1} v_{k_1} \rho_{k_2} v_{k_2} \sum_{\mathbf{j}} {}' (z - a_{k_1} - e_{\mathbf{j}})^{-2}\, \overline{E_2\left(\frac{r_{k_1}^2}{\overline{\overline{z - a_{k_1} - e_{\mathbf{j}}}}} + a_{k_1} - a_{k_2}\right)}$$

$$+\tfrac{2}{\pi^2}\textstyle\sum_{k_1=1}^N \rho_{k_1}^2 v_{k_1}^2 \sum_{\mathbf{j}}^* (z - a_{k_1} - e_{\mathbf{j}})^{-2}\, \overline{\sigma_2\left(\frac{r_{k_1}^2}{\overline{\overline{z - a_{k_1} - e_{\mathbf{j}}}}}\right)} + O(\rho^4), \text{ as } \rho \to 0.$$

We now proceed to calculate the value λ_e^y. It is sufficient to change α by α^{-1} and apply the formula (22). Let us consider the lattice $\mathcal{Q}^*$ defined by the fundamental translation vectors α^{-1} and $i\alpha$. By virture of (23) we have

$$\lambda_e^x + i\lambda_e^{xy} = 1 + 2\sum_{m=1}^N \rho_m v_m + \frac{S_2^*}{\pi}\sum_{m,k=1}^N \rho_m v_m \rho_k v_k + \frac{1}{\pi}\sum_{m\neq k}^N \rho_m v_m \rho_k v_k \mathcal{P}^*(a_k - a_m) +$$

$$\frac{2}{\pi^2}\sum_{m=1}^N \sum_{k_1=1}^N \sum_{k_2\neq k_1}^N \rho_m v_m \rho_{k_1} v_{k_1} \rho_{k_2} v_{k_2} \sum_{\mathbf{j}} {}^* (z - a_{k_1} - e_{\mathbf{j}})^{-2}\, \overline{E_2^*\left(\frac{r_{k_1}^2}{\overline{\overline{z - a_{k_1} - e_{\mathbf{j}}}}} + a_{k_1} - a_{k_2}\right)} \tag{24}$$

$$+\frac{2}{\pi^2}\sum_{k_1=1}^N \rho_{k_1}^2 v_{k_1}^2 \sum_{\mathbf{j}} {}^* (z - a_{k_1} - e_{\mathbf{j}})^{-2}\, \overline{\sigma_2^*\left(\frac{r_{k_1}^2}{\overline{\overline{z - a_{k_1} - e_{\mathbf{j}}}}}\right)} + O(\rho^4), \text{ as } \rho \to 0,$$

where S_2^*, $\mathcal{P}^*$, E_2^* and σ_2^* correspond to the lattice $\mathcal{Q}^*$. We try to calculate the parameters of $\mathcal{Q}^*$ by the parameters of $\mathcal{Q}$ changing $e_{\mathbf{j}}$ by $ie_{\mathbf{j}}$. It is easily seen that $\mathcal{P}^*(z) = \mathcal{P}(iz)$, $E_2^*(z) = E_2(iz)$, $\sigma_2^*(z) = \sigma_2(iz)$. So we need only to calculate S_2^* by S_2. It is known that [14, 15]

$$S_2(p) = \frac{\pi^2}{p}\left(\frac{1}{3} - 2\sum_{n=1}^\infty \sinh^{-2}(\pi p^{-1} m)\right) = \frac{\pi^2}{p}\left(\frac{1}{3} - 8\sum_{n=1}^\infty \frac{mh^{2m}}{1-h^{2m}}\right), \tag{25}$$

where p is the ratio of the sides of $Q_{(0,0)}$, $h := \exp(\pi p^{-1})$. Hence, we have $S_2 = S_2(\alpha^2)$ and $S_2^* = S_2(\alpha^{-2})$. We also prove the following relation

$$S_2(p) + S_2(p^{-1}) = 2\pi. \tag{26}$$

The direct proof of (26) by (25) is not known. Our proof is based on the following fundamental identity [5]

$$\lambda_e^x(\rho)\lambda_e^y(-\rho) = 1. \tag{27}$$

This formula relates the effective conductivity $\lambda_e^x(\rho)$ of the simple rectangular array of cylinders ($N = 1$) of conductivity $\lambda_1 = (1+\rho)/(1-\rho)$ to the effective conductivity $\lambda_e^y(-\rho)$ for the same array but with cylinders of conductivity λ_1^{-1}. The conductivity of the matrix is equal to unit in the both cases. We may take the simple rectangular array and prove (26) in this case, since $S_2(p)$ depends only on the ratio p. First we note that $S_2(p)$ is real number [14]. By (24) we have

$$\begin{aligned} \lambda_e^x(\rho) &= 1 + 2\rho v + 2\rho^2 v^2 S_2(p)/\pi + O(\rho^3), \\ \lambda_e^y(-\rho) &= 1 - 2\rho v + 2\rho^2 v^2 S_2(p^{-1})/\pi + O(\rho^3), \ \text{as } \rho \to 0, \end{aligned} \tag{28}$$

where $v = \pi r_1^2$. Substituting (28) into (27) and preserving terms up to $O(\rho^3)$ we obtain the relation (26).

6 Second order approximation and optimal design problems

In Sec.5 we obtain the exact formulae (22), (19) for the components of Λ_e. The third-order approximation (23) is deduced from (22). In the present section we use only second-order approximation. So (23), (24) take the form

$$\begin{aligned} \lambda_e^x - i\lambda_e^{xy} &= 1 + 2\sum_{m=1}^{N}\rho_m v_m + \frac{S_2}{\pi}\sum_{m,k=1}^{N}\rho_m v_m \rho_k v_k \\ &+ \frac{1}{\pi}\sum_{m\neq k}^{N}\rho_m v_m \rho_k v_k \mathcal{P}(a_k - a_m) + O(\rho^3), \ \text{as } \rho \to 0, \\ \lambda_e^y + i\lambda_e^{xy} &= 1 + 2\sum_{m=1}^{N}\rho_m v_m + \left(2 - \frac{S_2}{\pi}\right)\sum_{m,k=1}^{N}\rho_m v_m \rho_k v_k \\ &+ \frac{1}{\pi}\sum_{m\neq k}^{N}\rho_m v_m \rho_k v_k \mathcal{P}(i(a_k - a_m)) + O(\rho^3), \ \text{as } \rho \to 0. \end{aligned} \tag{29}$$

Rather than presenting Λ_e by (29), it is more useful to give the components of Λ_e along its major and minor axes (λ_{maj} and λ_{min}) and the angle θ between one of the principal axes and the x axis. The values $\lambda_{ma\mathbf{j}}$ and λ_{min} satisfy the square equation

$$(\lambda_e^x - \lambda)(\lambda_e^y - \lambda) - (\lambda_e^{xy})^2 = 0. \tag{30}$$

The angle θ has to satisfy the relation

$$\lambda_e^{xy}\left(\exp\left(i\theta\right)a_1, \exp\left(i\theta\right)a_2, ..., \exp\left(i\theta\right)a_N\right) = 0, \tag{31}$$

where $\lambda_e^{xy} = \lambda_e^{xy}\left(a_1, a_2, ..., a_N\right)$ is calculated by (23). Let us discuss the equalities (30) and (31) in the second-order approximation. Using (29) we obtain from (31) the follwing equation

$$\sum_{m\neq k}^{N} \rho_m v_m \rho_k v_k Im\, \mathcal{P}\left(\exp\left(i\theta\right)\left(a_k - a_m\right)\right) = 0 \tag{32}$$

with respect to θ.

Let us consider the case of the same inclusions, i.e. $\rho_k = \rho,\ v_k = v/N$ for each $k = 1, 2, ..., N$. Then (29) becomes

$$\begin{aligned} \lambda_e^x - i\lambda_e^{xy} &= 1 + 2\rho v + 2\rho^2 v^2 + \rho^2 v^2 \kappa + O(\rho^3), \\ \lambda_e^y + i\lambda_e^{xy} &= 1 - 2\rho v + 2\rho^2 v^2 - \rho^2 v^2 \kappa + O(\rho^3),\ \text{as } \rho \to 0, \end{aligned} \tag{33}$$

where

$$\kappa := 2\left(\frac{S_2}{\pi} - 1\right) + \frac{1}{\pi N^2}\sum_{m\neq k}^{N} \mathcal{P}(a_k - a_m).$$

We shall call the value κ by the anisotropy coefficient. In this case equation (32) becomes

$$Im \sum_{m\neq k}^{N} \mathcal{P}\left(\exp\left(i\theta\right)\left(a_k - a_m\right)\right) = 0. \tag{34}$$

Solving equation (30) in this case we obtain

$$\begin{aligned} \lambda_{maj} &= 1 + 2\rho v + 2\rho^2 v^2 + \rho^2 v^2 \left|\kappa\right| + O(\rho^3), \\ \lambda_{min} &= 1 - 2\rho v + 2\rho^2 v^2 - \rho^2 v^2 \left|\kappa\right| + O(\rho^3),\ \text{as } \rho \to 0. \end{aligned} \tag{35}$$

One can see that for the fixed ρ and v the value λ_{maj} attaches the maximum up to $O(\rho^3)$ (λ_{min} attaches the minimum) simultaneously with $|\kappa|$. We now prove this up to $O(\rho^4)$.

We consider the coefficient λ_e^{xy} in the third-order approximation:

$$\lambda_e^{xy} = \rho^2 v^2 \left(-Im\ \kappa + \rho v \frac{2}{\pi^2} Im\ \left(X + Y\right)\right) + o(\rho^3),\ \text{as } \rho \to 0,$$

where

$$X := \sum_{m=1}^{N}\sum_{k=1}^{N}\sum_{l\neq k}^{N} X_{mkl},\ X_{mkl} := \sum_{\mathbf{j}} {}^{\prime}(a_m - a_k - e_{\mathbf{j}})^{-2}\overline{E_2\left(\frac{r^2}{\overline{\overline{a_m - a_k - e_{\mathbf{j}}}}} + a_k - a_l\right)},$$

$$Y := \sum_{m=1}^{N}\sum_{k=1}^{N} Y_{mk},\ Y_{mk} := \sum_{\mathbf{j}} {}^{\prime}(a_m - a_k - e_{\mathbf{j}})^{-2}\overline{\sigma_2\left(\frac{r^2}{\overline{\overline{a_m - a_k - e_{\mathbf{j}}}}}\right)}$$

in accordance with (23). We have [18]

$$E_2(z_0 + \Delta z) = \sum_{n=0}^{\infty} E_2^{(n)}(z_0)\frac{1}{n!}(\Delta z)^n = \sum_{n=0}^{\infty} (-1)^n (n+1) E_{n-2}(z_0)(\Delta z)^n.$$

Then

$$\begin{aligned} X_{mkl} &= \sum_{\mathbf{j}}\sum_{n=0}^{\infty} (-1)^n r^{2n} (n+1) (a_m - a_k - e_{\mathbf{j}})^{-n-2}\overline{E_{n+2}(a_m - a_k)} = \\ &\sum_{n=0}^{\infty} (-1)^n r^{2n} (n+1) E_{n+2}(a_m - a_k)\,\overline{E_{n+2}(a_k - a_l)} \text{ for } k \neq m, \end{aligned}$$

and

$$X_{mml} = \sum_{n=0}^{\infty} (-1)^n r^{2n} (n+1) S_{n+2}\overline{E_{n+2}(a_m - a_l)}.$$

Similar

$$Y_{mk} = \sum_{n=0}^{\infty} r^{4(n-1)} (2n-1) S_{2n} E_{2n}(a_m - a_k) \text{ for } k \neq m,$$

$$Y_{mm} = \sum_{n=0}^{\infty} r^{4(n-1)} (2n-1) S_{2n}^2.$$

Using the properties of S_n wee have

$$X + Y = \sum_{m=1}^{N}\sum_{k\neq m}^{N}\sum_{l\neq k}^{N}\sum_{n=0}^{\infty} (-1)^n r^{2n} (n+1) E_{n+2}(a_m - a_k)\,\overline{E_{n+2}(a_k - a_l)}$$

$$\sum_{m=1}^{N}\sum_{k\neq m}^{N}\sum_{n=1}^{\infty} r^{4(n-1)} (2n-1) S_{2n}\left[\overline{E_{2n}(a_m - a_k)} + E_{2n}(a_m - a_k)\right] +$$

$$\sum_{m=1}^{N}\sum_{n=0}^{\infty} r^{4(n-1)}\,(2n-1)\,S_{2n}^{2}.$$

Calculating $\overline{X+Y}$ one can see that $\overline{X+Y} = X+Y$, hence $X+Y$ is real. Thus we have completely derived two-component composite material up to $O(\rho^4)$. In particular we have proved that (34), (35) are valid up $O(\rho^4)$. To take into account higher-order terms on ρ we need to study more complicated formulae (23). In particular formulae (34), (35) are corrected by higher-order terms on ρ.

In Fig. 1 - 4 we present numerical examples for the square array of cylinders, when $S_2 = \pi$ in the case $N = 3$.

7 Conclusions

The $\mathbb{R}$-linear problem (3) for a multiply connected circular domain in a class of doubly periodic functions has been solved in analytic form by a method of functional equations. Using this solution we have obtained the effective conductivity tensor of two-dimensional doubly periodic composite materials made from a collection of disks imbedded in an otherwise uniform host. The anisotropy coefficient has been introduced. This coefficient involves only geometrical parameters of the cell in consideration and derives anisotropic properties of the material in macroscale up to $O(\rho^4)$.

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Fig.1. The coordinates of the centers, $a_1 = 0,\ a_2 = 0.3,\ a_3 = 0.5 + 0.2i$. The calculated angle, $\theta = 0.76\frac{\pi}{2} = 1.114$; the modulus of the coefficient of anisotropy, $|\kappa| = 0.884$.

Fig.2. The coordinates of the centers, $a_1 = 0,\ a_2 = 0.5,\ a_3 = 0.3 + 0.6i$. The angle, $\theta = 0.3\frac{\pi}{2} = 0.471$. The modulus of the coefficient of anisotropy, $|\kappa| = 1.031$.

Fig.3. The coordinates of the centers, $a_1 = 0,\ a_2 = 0.5,\ a_3 = x + 0.6i$. The maximum modulus of the coefficient of anisotropy, $|\kappa| = 0.731$ (circle); the minimum, $|\kappa| = 0.584$ (dots).

Fig.4. The graphic of the function $|\kappa\,(x)|$, when $a_1 = 0,\ a_2 = 0.5,\ a_3 = x + 0.6i$.