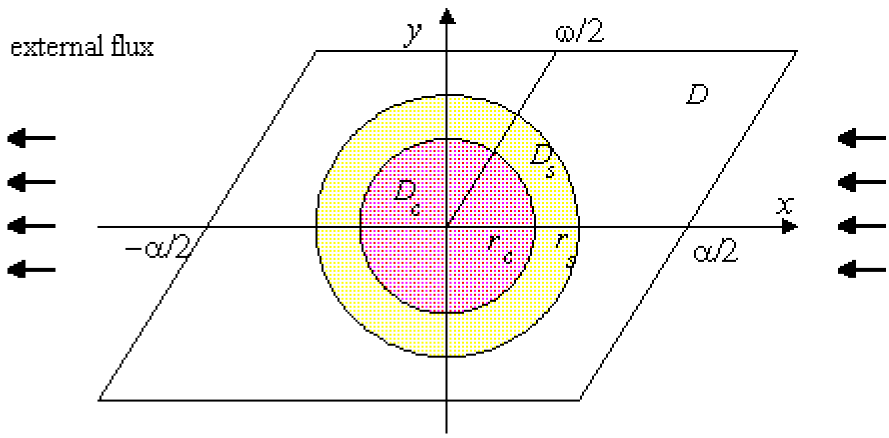
Transport Properties of a Regular Array of Coated Cylinders

V. Mityushev
Dept. Math., Pedagogical College, ul. Arciszewskiego 22 B, 76-200, Slupsk, Poland.

A regular array of coated circular cylinders is embedded in a homogeneous material. The present paper is devoted to calculation of the effective transport properties of this composite material. This work is particularly concerned with the thermal conductivity since the three - phase heat conduction problem describes the interface heat problem [1]. Due to mathematical equivalence, the results of the present work are applicable to the problems, whose transport phenomena are governed by the Laplace equation.

Background literature on the problem in question is represented in [1]. Much of the previous works is based on the method of Rayleigh [1-2], where the problem is reduced to an infinite system of linear algebraic equations, which can either be truncated to give various low-ordered formulae, or solved numerically on a computer. Moreover, the problem was considered by the self-consistent method, the method of finite elements and so on (see [1-2] and papers cited therein). The results [1-2] are restricted by the study of the square and hexagonal arrays of cylinders generating an isotropic material in macroscale. Despite of great success [1] numerical methods does not allow us to get exact or approximate general analytic formulae for the effective conductivity. Such formulae are very useful to optimize the properties of composite materials because they contain such parameters as conductivities of components, volume fraction, geometrical parameters of the lattice.

The present work is based on the functional equations method [3-6]. This method allows us to construct a simple algorithm to get approximate analytic formulae for the effective tensor for an arbitrary regular array of coated cylinders. In the cases of the square and hexagonal arrays we get the known formulae derived by other methods [1-2].



We study the conductivity of the doubly periodic composite material (see Fig.), when the domains D, D_C and D_S are occupied by materials of conductivity $\bullet_C$ and $\bullet_S$ respectively. Let the external field is applied in the x - direction. We find the potentials u, u_C and u_S to be harmonic in D, D_C and D_S with boundary conditions:

$$u = u_s, \frac{\partial u}{\partial n} = \lambda_s \frac{\partial u_s}{\partial n} \quad \text{on} \quad x^2 + y^2 = r_s^2, \quad u_s = u_c, \lambda_s \frac{\partial u_s}{\partial n} = \lambda_c \frac{\partial u_c}{\partial n} \quad \text{on} \quad x^2 + y^2 = r_c^2,$$

$$u(x+\alpha, y) - u(x,y) = \alpha, \ u(x+\beta, y+\alpha^{-1}) - u(x,y) = 0, \tag{1}$$

where $(\alpha,0),(\beta,\alpha^{-1})$ are the fundamental translation vectors, $\partial / \partial n$ is the normal derivative. The problem (1) is reduced to a functional equation, which implies, for instance, the formula for the effective conductivity tensor

$$\lambda_e^x + i\lambda_e^y \approx 1 + 2q + 2q^2 S_2/\pi - 2q\rho\sigma(r_c/r_s)^2 + 2q[(|S_2|/\pi)^2 q^2 - 2(S_2/\pi)\rho\sigma(r_c/r_s)^2 +$$

$$\rho^2\sigma^2(r_c/r_s)^4] + q\sum_{m=1}^{\infty}\left[\rho r_s^2 e_m^{-2}\overline{P(r_s^2/\overline{e_m})} + \sigma r_c^2 e_m^{-2}\overline{P(r_c^2/\overline{e_m})} - \rho r_s^2 - \sigma r_c^2\right].$$

Here $q := \rho\pi r_s^2 + \sigma\pi r_c^2$, $\rho := (\lambda_s - 1)/(\lambda_s + 1)$, $\sigma := (\lambda_c - \lambda_s)/(\lambda_c + \lambda_s)$, $P(z)$ is the Weierstrass function, S_2 is the Rayleigh sum [6], e_m are the centers of the cells except the zero one.

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