MACROSCOPIC CONDUCTIVITY OF CURVILINEAR CHANNELS

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ABSTRACT: Consider a channel with two-dimensional wavy walls whose amplitude is proportional to the mean clearance of the channel multiplied by a small dimensionless parameter ε. Using the method of perturbations we explicitly write the effective conductivity of the channel up to ε^3 for arbitrary shapes of the walls.

Introduction

Determination of the effective conductivity λ of curvilinear channels is an important applied problem. Though this problem can be solved by application of various numerical methods, it is interesting to obtain analytical formulae for the effective conductivity in order to find explicitly dependence on geometrical parameters. In the present paper, we consider a channel with two-dimensional wavy walls whose amplitude is proportional to the mean clearance of the channel multiplied by the small dimensionless parameter ε. Using the method described in [1], [2] we explicitly write λ up to $O(\varepsilon^3)$.

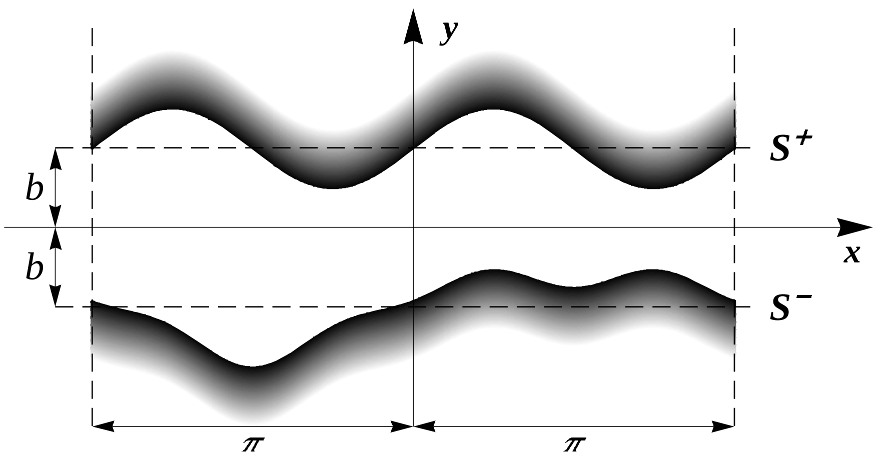

Figure 1: Bounded periodical channel domain D.

Let a periodical channel domain D is bounded by the top and bottom walls

$$S^{+}(x) \equiv b(1+\varepsilon T(x)), \quad S^{-}(x) \equiv b(-1+\varepsilon B(x)), \tag{0.1}$$

(Fig. 1) where $b > 0$ and ε is a non-negative dimensionless small parameter. It is assumed that the functions T and B are continuously differentiable and periodic in $[-\pi, \pi]$. The potential u satisfies the following problem

$$\begin{cases} \nabla^2 u(x,y) = 0, & (x,y) \in D, \\ u(\pi, y) - u(-\pi, y) = 2\pi, & \\ \frac{\partial u}{\partial \mathbf{n}}(x, S^{\pm}(x)) = 0. & \end{cases} \tag{0.2}$$

The second equation means that the potential has a constant jump along the x–axis. The third condition means that the normal flux on the surfaces vanishes. The solution of (0.2) can be found in the form [2]

$$u(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \varepsilon^3 u_3(x,y) + \ldots \tag{0.3}$$

In the simple case of the plane channel ($\varepsilon = 0$) the potential has the form $u_0(x,y) = x$.

All the following computations are performed with the accuracy $O(\varepsilon^2)$ which is noted by the asymptotic equality $\mathrel{\overset{\circ}{=}}$, in particular,

$$u(x,y) \mathrel{\overset{\circ}{=}} x + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y). \tag{0.4}$$

The normal vectors to the surfaces (0.1) have the form

$$\mathbf{n}^{+} = (-\varepsilon b T', 1), \quad \mathbf{n}^{-} = (\varepsilon b B', -1), \tag{0.5}$$

where primes denote the derivative. The normal derivatives of the potential with the required accuracy become

$$\frac{\partial u}{\partial \mathbf{n}^{+}} \mathrel{\overset{\circ}{=}} \varepsilon\left(-bT' + \frac{\partial u_1}{\partial y}\right) + \varepsilon^2\left(-bT'\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) = 0 \tag{0.6}$$

$$\frac{\partial u}{\partial \mathbf{n}^{-}} \mathrel{\overset{\circ}{=}} \varepsilon\left(bB' - \frac{\partial u_1}{\partial y}\right) + \varepsilon^2\left(bB'\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y}\right) = 0 \tag{0.7}$$

They are equal to zero because of the boundary conditions from (0.2). Take the coefficients on the same powers of ε in (0.6) and (0.7)

$$\begin{cases} -bT' + \frac{\partial u_1}{\partial y} = 0 \\ -bT'\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \end{cases} \tag{0.8}$$

$$\begin{cases} bB' - \frac{\partial u_1}{\partial y} = 0 \\ bB'\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y} = 0 \end{cases} \tag{0.9}$$

Consider now the problem (0.2) in the first order approximation

$$\begin{cases} \nabla^2 u_1 = 0, & (x,y) \in D_0 \\ u_1(\pi, y) - u_1(-\pi, y) = 0, & \\ \frac{\partial u_1}{\partial y}(x, b) = bT'(x), & \\ \frac{\partial u_1}{\partial y}(x, -b) = bB'(x) & \end{cases} \tag{0.10}$$

where $D_0 = \{(x,y) \in \mathbb{R}^2 : \ -b < y < b\}$.

The functions u_1, T and B can be presented as their complex Fourier series

$$T(x) = \sum_{\nu=-\infty}^{+\infty} T_\nu e^{i\nu x}, \quad B(x) = \sum_{\nu=-\infty}^{+\infty} B_\nu e^{i\nu x}, \tag{0.11}$$

$$u_1(x,y) = \sum_{\nu=-\infty}^{+\infty} c_\nu(y) e^{i\nu x}. \tag{0.12}$$

Then the conditions (0.10) can be written in the form

$$\nabla^2 u_1(x,y) = \sum_{\nu=-\infty}^{+\infty} (c_\nu''(y) - \nu^2 c_\nu(y)) e^{i\nu x} = 0 \tag{0.13}$$

$$\sum_{\nu=-\infty}^{+\infty} c_\nu'(b) e^{i\nu x} = b \sum_{\nu=-\infty}^{+\infty} i\nu T_\nu e^{i\nu x}, \quad \sum_{\nu=-\infty}^{+\infty} c_\nu'(-b) e^{i\nu x} = b \sum_{\nu=-\infty}^{+\infty} i\nu B_\nu e^{i\nu x} \tag{0.14}$$

Uniqueness of the Fourier representation yields the following ordinary differential equations and the boundary conditions for c_ν.

$$\begin{cases} c_\nu''(y) - \nu^2 c_\nu(y) = 0 \\ c_\nu'(b) = ib\nu T_\nu \\ c_\nu'(-b) = ib\nu B_\nu \end{cases} \tag{0.15}$$

The solution of (0.15) has the form

$$c_\nu(y) = \frac{ib}{\sinh(2\nu b)} \Big(T_\nu \cosh\big(\nu(b+y)\big) - B_\nu \cosh\big(\nu(b-y)\big) \Big). \tag{0.16}$$

1 Conductivity

The effective conductivity of the channel is defined as the double integral

$$\lambda_x = \frac{1}{|D|} \iint_D |\nabla u|^2 \, dxdy \tag{1.1}$$

It can be written in extended form as follows

$$\frac{1}{4\pi b} \int_{-\pi}^{\pi} dx \int_{S^-}^{S^+} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dy \tag{1.2}$$

It is easy to show that

$$\left(\frac{\partial u}{\partial x}\right)^2 \doteq 1 + 2\varepsilon \frac{\partial u_1}{\partial x} + \varepsilon^2 \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\varepsilon^2 \frac{\partial u_2}{\partial x} \tag{1.3}$$

and

$$\left(\frac{\partial u}{\partial y}\right)^2 \doteq \varepsilon^2 \left(\frac{\partial u_1}{\partial y}\right)^2. \tag{1.4}$$

Substitution of (1.3) and (1.4) into (1.2) yields

$$\lambda_x \doteq \frac{1}{4\pi b} \int_{-\pi}^{\pi} dx \int_{S^-}^{S^+} \left(1 + 2\varepsilon \frac{\partial u_1}{\partial x} + 2\varepsilon^2 \frac{\partial u_2}{\partial x} + \varepsilon^2 |\nabla u_1|^2 \right) dy. \tag{1.5}$$

One of the integrals of (1.5) is transformed as follows (with the accuracy $O(\varepsilon^2)$)

$$\frac{\varepsilon^2}{4\pi b}\iint_{D_0} |\nabla u_1|^2\, dxdy + 2b\int_{-\pi}^{\pi}\left(T\frac{\partial u_1}{\partial x}(x,b) - B\frac{\partial u_1}{\partial x}(x,-b)\right)dx, \tag{1.6}$$

since the functions u_1 and u_2 are periodic in x. Here, we use the following formula based on the Taylor theorem

$$\int_{S^-}^{S^+} f(y)dy \doteq \int_{-b}^{b} f(y)dy + b\varepsilon\left[Tf(b) - Bf(-b)\right] + \frac{b\varepsilon}{2}\left[Tf'(b) - Bf'(-b)\right]$$

Applycation of Green's theorem to the double integral in (1.6) and use of $\nabla^2 u_1 = 0$ yield

$$\iint_{D_0} |\nabla u_1|^2\, dxdy = \iint_{\partial D_0} u_1\frac{\partial u_1}{\partial n}ds \tag{1.7}$$

Compute (1.7) using the boundary and the periodicity conditions on ∂D_0

$$\iint_{D_0} |\nabla u_1|^2\, dxdy = b\int_{-\pi}^{\pi}(T'(x)u_1(x,b) - B'(x)u_1(x,-b))dx. \tag{1.8}$$

The latter integrals are calculated by parts. The periodicity of u_1, T and B yields

$$\int_{-\pi}^{\pi} T'(x)u_1(x,b)dx = -\int_{-\pi}^{\pi}\left(T(x)\frac{\partial u_1}{\partial x}(x,b)\right)dx \tag{1.9}$$

$$\int_{-\pi}^{\pi} B'(x)u_1(x,-b)dx = -\int_{-\pi}^{\pi} B(x)\frac{\partial u_1}{\partial x}(x,-b)dx \tag{1.10}$$

Hence,

$$\lambda_x = 1 + \frac{\varepsilon^2}{4\pi}\int_{-\pi}^{\pi} T(x)\frac{\partial u_1}{\partial x}(x,b) - B(x)\frac{\partial u_1}{\partial x}(x,-b)dx \tag{1.11}$$

The integral in (1.11) can be considered as the zeroth coefficient F_0 of the Fourier series of the integrand multiplied by 2π

$$2\pi F_0 = \int_{-\pi}^{\pi}\left[T(x)\frac{\partial u_1}{\partial x}(x,b) - B(x)\frac{\partial u_1}{\partial x}(x,-b)\right]dx \tag{1.12}$$

The zeroth coefficient of the product $T(x)$ and $\frac{\partial u_1}{\partial x}(x,b)$ takes the form

$$\left\{T(x)\frac{\partial u_1}{\partial x}(x,b)\right\}_0 = \sum_{\nu=-\infty}^{+\infty} T_{-\nu} i\nu c_\nu(b) = \sum_{\nu=-\infty}^{+\infty} -\nu b\overline{T_\nu}\frac{T_\nu\cosh(2\nu b) - B_\nu}{\sinh 2\nu b}, \tag{1.13}$$

where the bar denote the complex conjugation. Along similar lines

$$\left\{B(x)\frac{\partial u_1}{\partial x}(x,-b)\right\}_0 = \sum_{\nu=-\infty}^{+\infty} -\nu b\overline{B_\nu}\frac{T_\nu - B_\nu\cosh(2\nu b)}{\sinh 2\nu b} \tag{1.14}$$

Therefore,

$$\begin{aligned} F_0 &= \sum_{\nu=-\infty}^{+\infty}\frac{-\nu b}{\sinh(2\nu b)}\big(\overline{T_\nu}T_\nu\cosh(2\nu b) - \overline{T_\nu}B_\nu - \overline{B_\nu}T_\nu + \overline{B_\nu}B_\nu\cosh(2\nu b)\big) \\ &= -b\sum_{\nu=-\infty}^{+\infty}\frac{\nu}{\sinh(2\nu b)}\big(\cosh(2\nu b)(|T_\nu|^2 + |B_\nu|^2) - 2Re(\overline{T_\nu}B_\nu)\big), \end{aligned} \tag{1.15}$$

where Re stands for the real part. The ultimate formula for the effective conductivity becomes

$$\lambda_x = 1 - \frac{\varepsilon^2 b}{2} \sum_{\nu=-\infty}^{+\infty} \frac{\nu}{\sinh(2\nu b)} \big(\cosh(2\nu b)(|T_\nu|^2 + |B_\nu|^2) - 2Re(\overline{T_\nu} B_\nu)\big). \tag{1.16}$$

The results for the several bounded channels are presented in Fig. 2 and Fig. 3.

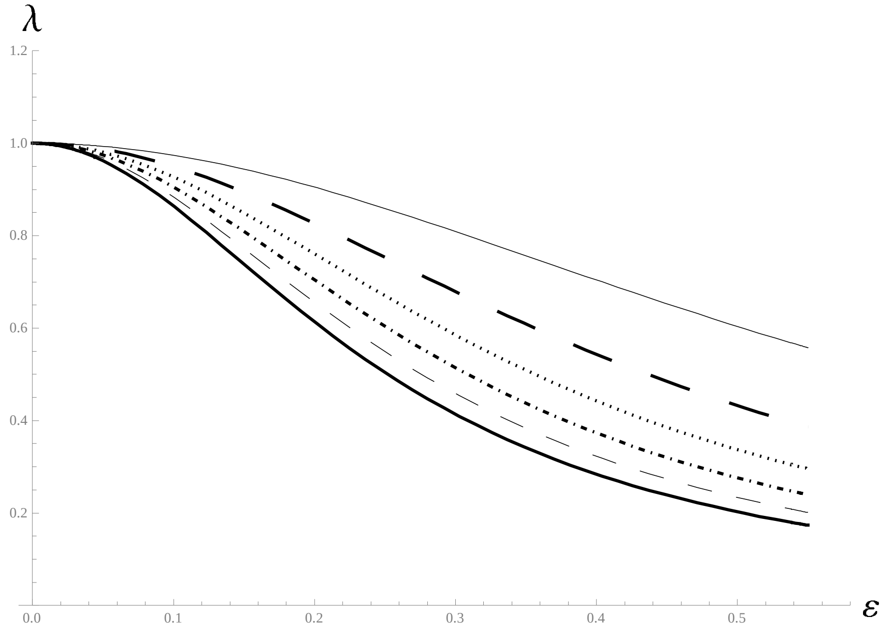

Figure 2: Effective conductivity calculated by (1.16) for $T(x) = -\cos(x)$, $B(x) = \sin(5x)$ and b from 1.5 to 9. Data are for: solid line: $b = 1.5$, thick broken line: $b = 3$; dots: $b = 4.5$; dots and broken line: $b = 6$; broken: $b = 7.5$; thick solid line: $b = 9$

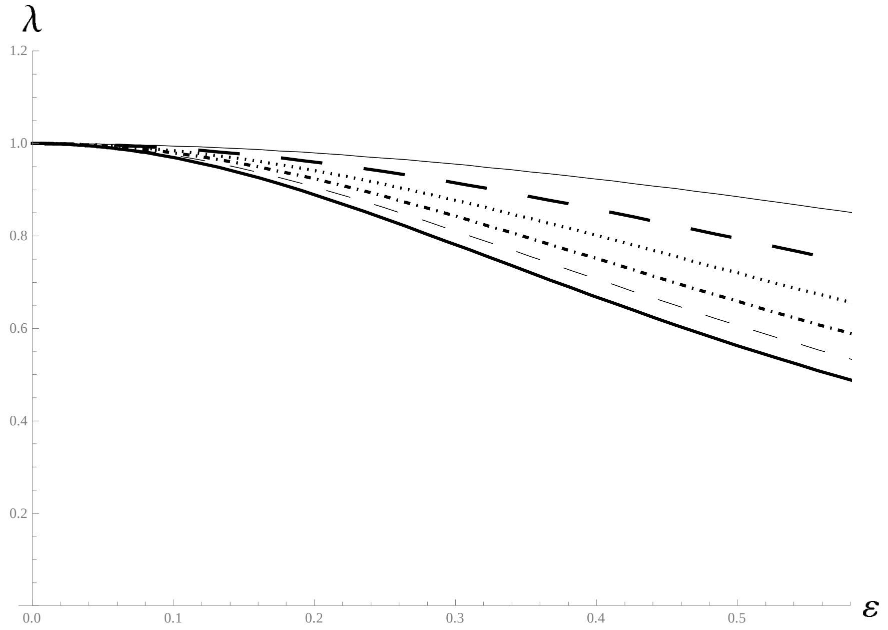

Figure 3: Effective conductivity calculated by (1.16) for $T(x) = 0$, $B(x) = \sin(x) + \frac{1}{3}\cos(2x) - \frac{1}{5}\cos(4x)$ and b from 1.5 to 9. Data are for: solid line: $b = 1.5$, thick broken line: $b = 3$; dots: $b = 4.5$; dots and broken line: $b = 6$; broken: $b = 7.5$; thick solid line: $b = 9$

References

[1] A. E. Malevich, V. V. Mityushev and P. M. Adler, Electrokinetic phenomena in wavy channels. J Colloid Interface Sci. 2010, 345(1), 72-87.

[2] A. E. Malevich, V. V. Mityushev and P. M. Adler, Stokes flow through a channel with wavy walls, Acta Mechanica, 2006, 182, 3-4, 151-182.