Transport properties of a fibre-layer composite material

V.V.Mityushev, T.N.Zhorovina

Abstract

We study a composite material of the fibre - layer type by combination of the method of symmetry and the method of functional equations. Approximate analytical formulae are deduced for the effective conductivity tensor.

1 Introduction

The transport properties of various types of two-dimensional composite materials have been studied by different mathematical methods. There are exact and approximate formulae for the effective conductivity tensor Λ_e for special types of the materials. The tensor Λ_e for the layer materials is calculated by the arithmetic and harmonic mean values of the properties of components. Exact formulae in the case of rectangular compounds arranged in chess order have been obtained in [9], [8]. If rectangular cells are square ones, we arrive at the Dykhne - Keller formula [1] involving the geometric mean value.

McPhedran *et al.* [2] extend the Rayleigh method to analyze the transport properties of composite materials consisting of a doubly periodic array of cylinders. See also papers of the present book. A rigorous justification of the Rayleigh lattice sums and relations between the Rayleigh and Schwarz methods are exposed in [4]. Exact and approximate formulae for Λ_e for simple double periodic array and complex double periodic arrays are deduced in [3], [6], [7] by the method of functional equations.

All these method are devoted to special types of composites: layer, chess, fibre. In the present paper we study a composite material of the fibre - layer type by combination of the method of symmetry and the method of functional equations. Similar problems for materials with finite number of inclusions are solved in [11].

2 Statement of the problem

We consider a lattice $\mathcal{Q}$ which is defined by two fundamental vectors α ($\alpha > 0$) and $i\alpha^{-1}$ on the complex plane $\mathbb{C}$ of the variable $z = x + iy$. The zero cell $Q_0 := \{z = t_1\alpha + t_2 i\alpha^{-1} : \ -1/2 < t_j < 1/2\}$ is divided onto the domains $D_3 := \{z \in Q_0 : \ |z - ia| < r\}$, $D_1 := \{z \in Q_0 \setminus (D_3 \cup L) : \ \Im z > 0\}$, $D_4 := \{z \in Q_0 : \ |z + ia| < r\}$, $D_2 := \{z \in Q_0 \setminus (D_4 \cup \Gamma) : \ \Im z < 0\}$, where $L := \partial D_3$, $\Gamma := \partial D_4$. We denote the boundary of D_j by ∂D_j. See Figure.

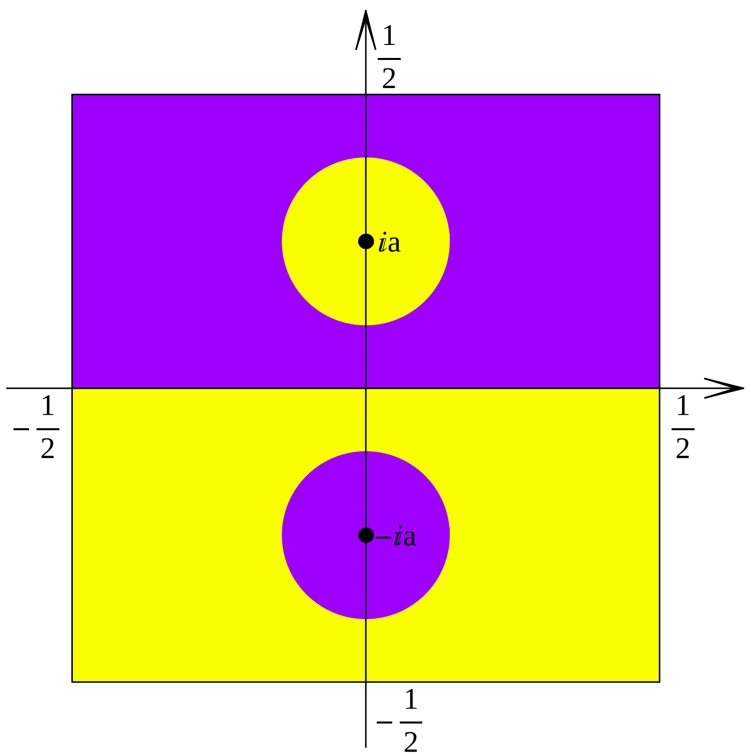

Figure 1:

Let a material of the conductivity λ_1 occupies D_1 and D_4, a material of the conductivity λ_2 occupies D_2 and D_3. The external field is applied in the direction of the negative x axis. The general theory implies the following boundary value problem. To find functions $u_j(z)$ harmonic in D_j respectively ($j = 1, 2, 3, 4$) and continuously differentiable in $(D_j \cup \partial D_j) \setminus \{\pm\alpha/2\}$; $u_j(z)$ for $j = 1, 2$ are bounded near the points $\pm\alpha/2$. The unknown functions satisfy the boundary conditions:

$$u_j(\frac{\alpha}{2} + iy) - u_j(-\frac{\alpha}{2} + iy) = \alpha, \ 0 < |y| < \frac{1}{2\alpha},$$

$$\frac{\partial u_j}{\partial x}(\frac{\alpha}{2}+iy)-\frac{\partial u_j}{\partial x}(-\frac{\alpha}{2}+iy),\ 0<|y|<\frac{1}{2\alpha},\ j=1,2;$$

$$u_1(x+\frac{i}{2\alpha})=u_2(x-\frac{i}{2\alpha}),\ \lambda_1\frac{\partial u_1}{\partial y}(x+\frac{i}{2\alpha})=\lambda_2\frac{\partial u_2}{\partial y}(x-\frac{i}{2\alpha}),\ |x|<\frac{\alpha}{2},\tag{2.1}$$

$$u_1(x+i0)=u_2(x-i0),\ \lambda_1\frac{\partial u_1}{\partial y}(x+i0)=\lambda_2\frac{\partial u_2}{\partial y}(x-i0),\ |x|<\frac{\alpha}{2},$$

$$u_1=u_3,\ \lambda_1\frac{\partial u_1}{\partial n}=\lambda_2\frac{\partial u_3}{\partial n}\text{ on }L,$$

$$u_2=u_4,\ \lambda_2\frac{\partial u_2}{\partial n}=\lambda_1\frac{\partial u_4}{\partial n}\text{ on }\Gamma,$$

where $\partial/\partial n$ is the normal derivative outward to L and Γ.

Following [3], [6], [7] we introduce the complex potentials $\varphi_j(z)$ analytic in D_j in such a way that $\Re\varphi_j(z)=u_j(z)$ in D_j. Then the conditions (2.1) become

$$\varphi_j(t+\alpha)-\varphi_j(t)=\alpha+i\gamma,\ t=-\frac{\alpha}{2}+iy,\ 0<|y|<\frac{1}{2\alpha},\ j=1,2,$$

$$\varphi_1(t)=\frac{\lambda_1+\lambda_2}{2\lambda_1}\varphi_2(t)+\frac{\lambda_1-\lambda_2}{2\lambda_1}\overline{\varphi_2(t)},\ t=x,\ |x|<\frac{\alpha}{2},$$

$$\varphi_1(t)=\frac{\lambda_1+\lambda_2}{2\lambda_1}\varphi_3(t)+\frac{\lambda_1-\lambda_2}{2\lambda_1}\overline{\varphi_3(t)},\ t\in L,\tag{2.2}$$

$$\varphi_1(t+\frac{i}{\alpha})=\frac{\lambda_1+\lambda_2}{2\lambda_1}\varphi_2(t)+\frac{\lambda_1-\lambda_2}{2\lambda_1}\overline{\varphi_2(t)},\ t=x-\frac{i}{2\alpha},\ |x|<\frac{\alpha}{2},$$

$$\varphi_2(t)=\frac{\lambda_2+\lambda_1}{2\lambda_2}\varphi_4(t)+\frac{\lambda_2-\lambda_1}{2\lambda_2}\overline{\varphi_4(t)},\ t\in\Gamma,$$

where γ is a real constant. Introduce the vector-functions

$$\Phi(z)=\begin{pmatrix}\Phi_1(z)\\ \Phi_2(z)\end{pmatrix}:=\begin{pmatrix}\varphi_1(z)\\ \overline{\varphi_2(\overline{z})}\end{pmatrix},\ z\in D_1;\ \Theta(z)=\begin{pmatrix}\varphi_2(z)\\ \overline{\varphi_1(\overline{z})}\end{pmatrix},\ z\in D_2;$$

$$V(z):=\begin{pmatrix}\frac{\lambda_1+\lambda_2}{2\lambda_1}\varphi_3(z)\\ \frac{\lambda_1+\lambda_2}{2\lambda_2}\overline{\varphi_4(\overline{z})}\end{pmatrix},\ z\in D_3,\ V(z):=\begin{pmatrix}\varphi_4(z)+\frac{\lambda_1-\lambda_2}{2\lambda_1}\overline{\varphi_3(\overline{z})}\\ \overline{\varphi_3(\overline{z})}-\frac{\lambda_1-\lambda_2}{2\lambda_2}\varphi_4(z)\end{pmatrix},\ z\in D_4.$$

$V(z)$ is defined in different domains, but the same symbol is used for convenience. For instance, the second equality (2.2) can be written through the components of the vector-functions Φ and Θ as follows

$$\Phi_1(t)=\frac{\lambda_1+\lambda_2}{2\lambda_1}\Theta_1(t)+\frac{\lambda_1-\lambda_2}{2\lambda_1}\Phi_2(t),\tag{2.3}$$

$$\Theta_2(t) = \frac{\lambda_1+\lambda_2}{2\lambda_1}\Phi_2(t) + \frac{\lambda_1-\lambda_2}{2\lambda_1}\Theta_1(t),\ t = x,\ |x| < \frac{\alpha}{2}. \tag{2.4}$$

One can see that (2.4) is the complex conjugation of (2.3). Her the relation $\overline{t} = t$ on the real axis is used. The conditions (2.3), (2.4) can be written in the vector-matrix form

$$\Phi(t) = G\Theta(t),\ t = x,\ |x| < \frac{\alpha}{2}, \tag{2.5}$$

where

$$G = \frac{1}{\lambda_1+\lambda_2}\left(\begin{array}{cc} 2\lambda_2 & \lambda_1-\lambda_2 \\ -(\lambda_1-\lambda_2) & 2\lambda_1 \end{array}\right).$$

We introduce a new vector-function $F(z) = (F_1(z), F_2(z))^T$ as follows

$$F(z) := \left\{\begin{array}{c} \Phi(z),\ z \in D_1, \\ G\Theta(z),\ z \in D_2. \end{array}\right.$$

Then (2.5) in terms of $F(z)$ takes the form of the analytic continuation $F(x+i0) = F(x-i0),\ |x| < \frac{\alpha}{2}$. Similar arguments applied to the forth relation (2.2) yield the relation $F(x+\frac{i}{2\alpha}) = F(x-\frac{i}{2\alpha}),\ |x| < \frac{\alpha}{2}$. The third and fifth conditions (2.2) yield

$$F(t) = V(t) + G_1\overline{V(t)},\ t \in L,\ F(t) = V(t) + G_2\overline{V(t)},\ t \in \Gamma, \tag{2.6}$$

where

$$G_1 = \rho\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right),\ G_2 = \rho\left(\begin{array}{cc} 2\rho^2-1 & 2\rho(1-\rho) \\ 2\rho(1+\rho) & -(2\rho^2-1) \end{array}\right),\ \rho := \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}.$$

The first relation (2.2) implies

$$F(t+\alpha) - F(t) = \alpha\mathbf{1} + i\gamma B(t),\ t = -\frac{\alpha}{2} + iy,\ 0 < |y| < \frac{1}{2\alpha}, \tag{2.7}$$

where $\mathbf{1} := (1,1)^T,\ B(z) := (1,-1)^T,\ z \in D_1, B(z) := (\lambda_1+\lambda_2)^{-1}(3\lambda_2-\lambda_1, 3\lambda_1-\lambda_2)^T$, $z \in D_2$.

Thus we have the $\mathbb{R}-$linear problem (2.6) in the cell Q_0 with respect to the vector-functions $F(z),\ V(z)$ analytic in $D := Q_0\backslash(D_3 \cup L \cup D_4 \cup \Gamma),\ D_3 \cup D_4$, respectively. $F(z)$ is periodic along the y axis and has the prescribed jump along the x axis. This jump is derived by (2.7). Differentiating (2.6) we arrive to the $\mathbb{R}-$linear problem

$$F'(t) = V'(t) - \left(\frac{r}{t-ia}\right)^2 G_1\overline{V'(t)},\ t \in L,\ F'(t) = V'(t) - \left(\frac{r}{t+ia}\right)^2 G_2\overline{V'(t)},\ t \in \Gamma, \tag{2.8}$$

where the vector-function $F'(z)$ is doubly periodic.

3 Functional equations

The vector-matrix problem (2.6), (2.7) is similar to a scalar $\mathbb{R}-$linear problem solved in [6]. It is possible to repeat the arguments of [6] and deduce a system of functional equations. In order to write this system we introduce some objects. In particular we shall use the Eisenstein functions [10]

$$E_l(z) := \sum_{m_1,m_2=-\infty}^{+\infty} (z - e_{m_1,m_2})^{-l}, \ \sigma_l(z) = E_l(z) - z^{-l}, \ l = 2, 3, ..., \quad (3.1)$$

where $e_{m_1,m_2} := m_1\alpha + im_2\alpha^{-1}$. The series (3.1) for $l \geq 3$ converges absolutely and uniformly in each compact subset of the complex plane except the points e_{m_1,m_2}. If $l = 2$ then $E_2(z) := \mathcal{P}(z) + S_2, \ S_2 := 2\alpha^{-1}\zeta\alpha/2)$, where $\mathcal{P}(z)$ and $\zeta(z)$ are the Weierstrass functions [10].

Consider the Banach space $\mathcal{C}_L$ consisting of vector-functions continuous on L with the norm $\|\psi\| := \max_L \max_j |\psi_j(t)|$, where $\psi_j(t)$ is the $j-$th coordinate of $\psi(t)$. Introduced the closed subspace $\mathcal{C}_L^+ \subset \mathcal{C}_L$ of the vector-functions analytically continued into D_3. Analogously the spaces $\mathcal{C}_\Gamma^+$ and $\mathcal{C}_\Gamma$ are introduced. We also use the space $\mathcal{C}^+$ containing the vector-functions $\psi \in \mathcal{C}_L^+ \cap \mathcal{C}_\Gamma^+$. Actually $\mathcal{C}^+ \cong \mathcal{C}_L^+ \times \mathcal{C}_\Gamma^+$, since domains of the functions of $\mathcal{C}_L^+$ and $\mathcal{C}_\Gamma^+$ have not joint points.

The vector-function $V'(z)$ from the problem (2.8) belongs to $\mathcal{C}^+$. Hence, it can be represented in the form of its Taylor expansion

$$V'(z) = \begin{cases} \sum_{l=0}^{\infty} \alpha_l (z - ia)^l, & z \in D_3, \\ \sum_{l=0}^{\infty} \beta_l (z - ia)^l, & z \in D_4, \end{cases}$$

where α_l and β_l are two-dimensional vectors. Introduce the operators

$$\mathbf{W}V'(z) := \sum_{l=0}^{\infty} r^{2(l+1)} \left[G_1 \overline{\alpha_l} E_{l+2}(z - ia) + G_2 \overline{\beta_l} E_{l+2}(z + ia) \right], \ z \in D,$$

$$\mathbf{W}_L V'(z) := \sum_{l=0}^{\infty} r^{2(l+1)} \left[G_1 \overline{\alpha_l} \sigma_{l+2}(z - ia) + G_2 \overline{\beta_l} E_{l+2}(z + ia) \right], \ z \in Q_0 \backslash D_1,$$

$$(3.2)$$

$$\mathbf{W}_\Gamma V'(z) := \sum_{l=0}^{\infty} r^{2(l+1)} \left[G_1 \overline{\alpha_l} E_{l+2}(z - ia) + G_2 \overline{\beta_l} \sigma_{l+2}(z + ia) \right], \ z \in Q_0 \backslash D_3.$$

It follows from [5] that the series (3.2) converges absolutely and uniformly in the closures of the domains where they are defined. This implies that they are analytic in these domains. The function $\mathbf{W}V'(z)$ is doubly periodic, since it is a linear combination of the doubly periodic functions $E_l(z)$, $l = 2, 3, ...$.

We introduce the vector-function

$$\Xi(z) := \begin{cases} V'(z) + \mathbf{W}_L V'(z), & |z - ia| < r, \\ V'(z) + \mathbf{W}_\Gamma V'(z), & |z + ia| < r, \\ F'(z) + \mathbf{W} V'(z) & z \in D. \end{cases}$$

The jump of $\Xi(z)$ along $|t - ia| = r$ is calculated as follows

$$F'(t) + \mathbf{W}V'(t) - V'(t) - \mathbf{W}_L V'(t) = F'(t) - V'(t) + \left(\frac{r}{t - ia}\right)^2 G_1 \overline{V'(t)} = 0.$$

Here (2.8) and the relation $\frac{r^2}{t-ia} + ia = t$ on $|t - ia| = r$ are used. Analogously we obtain the zero jump of $\Xi(z)$ along $|t + ia| = r$. Applying the principle of analytic continuation and the general Liouville theorem for the lattice $\mathcal{Q}$ we conclude that $\Xi(z)$ is a constant vector. Following [3] it is possible to show that this constant vector is $\mathbf{1}$. The definition of $\Xi(z)$ implies the system of functional equations

$$V'(z) := \sum_{l=0}^{\infty} r^{2(l+1)} \left[G_1 \overline{\alpha_l} \sigma_{l+2}(z - ia) + G_2 \overline{\beta_l} E_{l+2}(z + ia)\right] + \mathbf{1}, \ |z - ia| < r,$$
$$V'(z) := \sum_{l=0}^{\infty} r^{2(l+1)} \left[G_1 \overline{\alpha_l} E_{l+2}(z - ia) + G_2 \overline{\beta_l} \sigma_{l+2}(z + ia)\right] + \mathbf{1}, \ |z + ia| < r. \tag{3.3}$$

Theorem 3.1. . *The system (3.3) has a unique solution in $\mathcal{C}^+$. This solution can be found by the method of successive approximations.*

The proof of the theorem repeats analogous proof of the scalar theorem from [5].

4 Effective conductivity

We apply Theorem 3.1 to calculate the component λ_e^x of the effective properties tensor

$$\Lambda_e = \begin{pmatrix} \lambda_e^x & 0 \\ 0 & \lambda_e^y \end{pmatrix}$$

of the composite material represented by the zero cell Q_0. We have

$$\lambda_e^x = \lambda_1 J_1 + \lambda_2 J_2, \tag{4.1}$$

where

$$J_1 = \int\int_{D_1} \frac{\partial u_1}{\partial x} dxdy + \int\int_{D_4} \frac{\partial u_4}{\partial x} dxdy, \ J_2 = \int\int_{D_2} \frac{\partial u_2}{\partial x} dxdy + \int\int_{D_3} \frac{\partial u_3}{\partial x} dxdy.$$

Applying Green's formula $\int\int_D \frac{\partial Q}{\partial x} dxdy = \int_D Qdy$ we obtain

$$J_1 = \int_{\partial D_1} u_1 dy + \int_\Gamma u_4 dy = \frac{1}{2} - \int_L u_3 dy + \int_\Gamma u_4 dy,$$

since $u_1 = u_3$ on L and the jump of u_1 on the opposite vertical sides of Q_0 is equal to α, the length of these sides is equal to $\alpha^{-1}/2$. It follows from the mean value theorem of harmonic functions that

$$J_1 = \frac{1}{2} + \pi r^2 \left[\frac{\partial u_4}{\partial x}(ia) - \frac{\partial u_3}{\partial x}(-ia) \right]. \quad (4.2)$$

Analogously

$$J_2 = \frac{1}{2} + \pi r^2 \left[\frac{\partial u_4}{\partial x}(-ia) - \frac{\partial u_3}{\partial x}(ia) \right]. \quad (4.3)$$

Substituting (4.2), (4.3) in (4.1) we obtain

$$\lambda_e^x = \lambda_0^x + \pi r^2 \left(\lambda_1 - \lambda_2\right) \left[\frac{\partial u_4}{\partial x}(-ia) - \frac{\partial u_3}{\partial x}(ia) \right], \quad (4.4)$$

where $\lambda_0^x = \left(\lambda_1 + \lambda_2\right)/2$ is the effective conductivity of the layer material, when the inclusions D_3 and D_4 are absent ($r = 0$).

Let us use in (4.4) the complex potentials. We have

$$\varphi_j'(z) = \frac{\partial u_j}{\partial x}(x, y) - i \frac{\partial u_j}{\partial y}(x, y),$$

then $\Re \varphi_j'(z) = \frac{\partial u_j}{\partial x}(x, y)$. In accordance with the definitions of $V'(z)$ and complex potentials the following relation

$$\begin{pmatrix} \frac{\partial u_3}{\partial x}(ia) \\ \frac{\partial u_4}{\partial x}(-ia) \end{pmatrix} = \frac{2}{\lambda_1 + \lambda_2} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Re V'(ia)$$

holds. Then (4.4) yields

$$\lambda_e^x = \lambda_0^x + 2\pi r^2 \rho \left[\lambda_2 V_1'(ia) - \lambda_1 V_2'(ia) \right], \quad (4.5)$$

where $V_1'(ia)$ and $V_2'(ia)$ are the components of the vector $V'(ia)$.

Thus we can propose the following constructive method to calculate λ_e^x. First we solve (3.3) by the method of successive approximations. Further, we calculate $\Re V'(ia)$ and substitute it in (4.5). Let us consider some simple analytic formulae which follow from this method. We have in the first order approximation the following formulae

$$\lambda_e^x = \lambda_0^x \left[1 - 4\pi r^2 \rho^2 - 4\pi r^4 \Re \mathcal{P}(2ia) \rho^3 (2\rho + 1) \right] + O(\rho^4), \text{ as } \rho \to 0, \quad (4.6)$$

where $\mathcal{P}(z)$ is the Weierstrass function. We shall use the Keller identity $\lambda_e^x \lambda_e^y = \lambda_1 \lambda_2$. Let us note that if we change λ_1 and λ_2, then λ_e^x and λ_e^y do not change because of symmetry. We have

$$\lambda_e^y = \lambda_0^y \left[1 + 4\pi r^2 \rho^2 + 4\pi r^4 \Re \mathcal{P}(2ia)\rho^3(2\rho+1)\right] + O(\rho^4), \tag{4.7}$$

where $\lambda_0^y = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}$. One can apply the next approximations to (3.3) and deduce higher-order formulae. It is possible to apply the method of successive approximations to (3.3) assuming that r^2 is a small parameter and deduce formulae with small r^2 and arbitrary ρ.

References

[1] Berdichevskij V L (1986) Variational methods of continuum mechanics, Moscow, Nauka

[2] McPhedran R C (1986) Transport properties of cylinder pairs and the square array of cylinders. Proc. Roy. Soc. Lond. A408: 31-43

[3] Mityushev V (1997) Transport properties of regular arrays of cylinders. ZAMM 77: 115-120

[4] Mityushev V (1997) Transport properties of finite and infinite composite materials and Rayleigh's sum. Arch.Mech. 49: 345-358

[5] Mityushev V (1997) A functional equation in a class of analytic functions and composite materials. Demostratio Math 30: 63-70

[6] Mityushev V (1996) Application of functional equations to the effective thermal conductivity of composite materials. WSP Publisher, Slupsk [in Polish]

[7] Mityushev V (1998) Steady heat conduction of the material with an array of cylindrical holes in the nonlinear case, IMA J Appl. Math. 61: 91-102

[8] Mityushev V, Zhorovina T N (1996) Exact solution to the problem on electrical field in doubly periodic heterogeneous media. Proc. Int. Conf. Boundary Value Problems, Special Functions and Fractional Calculus, Minsk, Belarus, February 16-20, 1996, 237-243 [in Russian]

[9] Obnosov Yu. (1996) Exact solution of a boundary value problem for a rectangular checkboard field. Proc. Roy. Soc. Lond. A452: 2423-2442

[10] Weil A (1976) Elliptic functions according to Eisenstein and Kronecker, Springer-Verlag, Berlin etc.

[11] Mityushev V, Zhorovina T N (1991) Unsteady heat conduction of certain composite material. Ing. Phys.J. 61: 866-867 [in Russian]