DIRICHLET PROBLEM WITH PRESCRIBED VORTICES FOR MULTIPLY CONNECTED DOMAINS

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The Dirichlet problem with prescribed vortices for the two-dimensional Laplace equation in multiply connected domains is reduced to the Riemann-Hilbert boundary value problem. This problem can be solved by iterative functional equations for circular domains. The solution of functional equations is derived in terms of the uniformly convergent Poincaré series. The obtained solution can be applied to the minimization of the Ginzburg-Landau functional.

KEY WORDS: multiply connected domain, Dirichlet problem, Ginzburg-Landau functional

MSC (2000): 30E25, 30D05

1 INTRODUCTION

One of the fundamental result of the minimisation problem for the two-dimensional Ginzburg-Landau functional is based on the observation that its minimum is expressed through a solution of the linear boundary value problem [1] when a potential satisfies Laplace's equation and has prescribed vortices on holes of the multiply connected domain. The present paper is devoted to solution to such a boundary value problem for multiply connected circular domains bounded by mutually disjoint circles on the complex plane (see Fig.1). The considered problem is reduced to the general Riemann-Hilbert boundary value problem solved in [6], [8]. The crucial point in solution is reduction of the problem to iterative functional equations for analytic functions. Application of successive iterations to the functional equations yields the famous Poincaré θ_2–series associated to the Schottky group [7]. Iterative functional equations in classes of analytic functions were discussed in [5]; see also extended review in the book [8].

In the present paper, we discuss the Dirichlet problem for multiply connected circular domains in a class of functions having prescribed vortices. We follow the papers [9], [10] where the canonical conformal mappings were constructed via the Riemann-Hilbert problem with a prescribed singularity. In the present paper, such a singularity is absent but the prescribed vortices yield the "β-part"of the solution (see formula (45) from [9]). The study of [9] concerned the "μ-part"with the vanishing "β-part"in the final formulae. In the present paper, the "μ-part"is absent at the beginning in the statement of the problem and the "β-part"is investigated.

Let d_k be given real constants such that $\sum_{k=1}^{n} d_k = 0$. Let $\partial/\partial n$ denote the outward normal derivative to L_k, where $L_k = \{z \in \mathbb{C} : \ |z - a_k| = r_k\}$ is positively oriented, i.e., leaves the mutually disjoint disk $\mathbb{D}_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$ $(k = 1,2,\dots,n)$ on the left. Consider the following problem for the function $U(z)$ continuously differentiable in the closure of the multiply connected domain $\mathbb{D} = \{z \in \mathbb{C} \cup \{\infty\} : |z - a_k| > r_k,\ k = 1,2,\dots,n\}$:

$$\begin{cases} \Delta U(z) = 0, \quad z \in \mathbb{D}, \\ U(t) = c_k, \quad t \in L_k, \\ U(\infty) = 0, \\ \frac{1}{2\pi}\int_{L_k} \frac{\partial U}{\partial n} ds = d_k, \quad (k = 1,2,\dots,n), \end{cases} \tag{1.1}$$

where Δ stands for the Laplace operator, the constants c_k are undetermined and have to be found during solution to the problem.

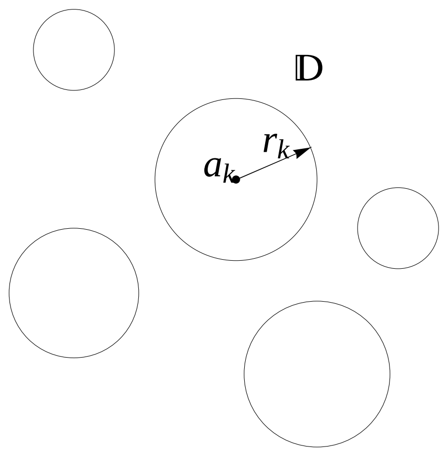

Рис. 1: Multiply connected circular domain $\mathbb{D}$.

This problem (1.1) generalizes the modified Dirichlet problem [4], [8] when $d_k = 0$ $(k = 1,2,\dots,n)$ and has the following relation to the Ginzburg-Landau functional [1]. Let $H^1(\mathbb{D};S^1)$ denote the Sobolev space of functions defined in $\mathbb{D}$ and having its values on the unit circle S^1 of the complex plane $\mathbb{C}$. Consider the class of maps

$$V = \{v \in H^1(\mathbb{D};S^1) : \deg(v, L_k) = d_k\}, \tag{1.2}$$

where $\deg(v, L_k)$ stands for the Brouwer degree, i.e., the winding number of v along the curve L_k. The energy functional introduced in [1] has the form

$$E[v] = \frac{1}{2}\int\limits_{\mathbb{D}} |\nabla v|^2 dx_1 dx_2, \tag{1.3}$$

where $z = x_1 + ix_2$ and i denotes the imaginary unit. It is proved in [1] that

$$\inf_{v \in V} E[v] = \frac{1}{2}\int\limits_{\mathbb{D}} |\nabla U|^2 dx_1 dx_2, \tag{1.4}$$

where U is a solution of the problem (1.1). This solution is unique up to an arbitrary additive constant and U minimizes the functional

$$F[v] = \frac{1}{2}\int_{\mathbb{D}} |\nabla v|^2 dx_1 dx_2 + 2\pi \sum_{k=1}^{n} d_k v|_{L_k} \tag{1.5}$$

in the class $\{v \in H^1(\mathbb{D};\mathbb{R}) : v|_{L_k} = g_k\}$, where g_k are constants.

2 BOUNDARY VALUE PROBLEMS

Consider the boundary value problem (1.1). The function $U(z)$ as a function harmonic in the multiply connected domain $\mathbb{D}$ can be presented in the form [8]

$$U(z) = \mathrm{Re}\,\varphi(z), \tag{2.1}$$

where

$$\varphi(z) = \phi(z) + \sum_{k=1}^{n} A_k \ln(z - a_k). \tag{2.2}$$

Here, the real constants A_k have the zero sum:

$$\sum_{k=1}^{n} A_k = 0. \tag{2.3}$$

The functions $\varphi(z)$ and $\phi(z)$ are analytic in $\mathbb{D}$ and continuously differentiable in the closure of $\mathbb{D}$, $\phi(z)$ is single-valued. A branch of the logarithm is arbitrary fixed. It does not impact on the result (2.2) since $U(z)$ depends only on

$$\sum_{k=1}^{n} A_k \ln|z - a_k|.$$

The functions $\varphi(z)$ and $\phi(z)$ vanish at infinity, i.e. $\varphi(\infty) = \phi(\infty) = 0$.

We now demonstrate that $A_k = d_k$ in (2.2). Let $\varphi(z) = U(z) + iV(z)$, where $U(z)$ and $V(z)$ stand for the real and imaginary parts of $\varphi(z)$. Let s denote the natural parameter of L_k. It is related with the complex coordinate $t \in L_k$ by formula

$$t = a_k + r_k \exp\left(\frac{is}{r_k}\right). \tag{2.4}$$

The Cauchy-Riemann equations imply [3] that

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial s}. \tag{2.5}$$

Calculate the integral

$$\frac{1}{2\pi}\int_{L_k} \frac{\partial U}{\partial n} ds = \frac{1}{2\pi}\int_{L_k} \frac{\partial V}{\partial s} ds = \frac{1}{2\pi}[V]_{L_k}, \tag{2.6}$$

where $[V]_{L_k}$ denotes the increment of V along L_k. Equation (2.2) yields

$$\frac{1}{2\pi}[V]_{L_k} = A_k, \tag{2.7}$$

since $\phi(z)$ is single-valued and

$$[\ln(z-a_k)]_{L_k} = 2\pi i.$$

The last equation of (1.1) and (2.6)-(2.7) yield $A_k = d_k$ $(k = 1,2,\dots,n)$ in the representation (2.2).

Using (2.4) we can calculate the differentials

$$dt = i\,\frac{t-a_k}{r_k}ds, \quad d\bar{t} = -i\,\frac{\overline{t-a_k}}{r_k}ds, \quad t \in L_k, \tag{2.8}$$

where the bar denotes the complex conjugation. Using (2.2) we can write the boundary condition expressed by the second equation of (1.1) in terms of the analytic function

$$\varphi(t) + \overline{\varphi(t)} = 2c_k, \quad |t-a_k| = r_k \quad (k = 1,2,\dots,n). \tag{2.9}$$

One may differentiate the boundary conditions (2.9) on the natural parameter s. Application of (2.8) yields

$$\frac{t-a_k}{r_k}\psi(t) - \frac{\overline{t-a_k}}{r_k}\overline{\psi(t)} = 0, \quad t \in L_k\ (k = 1,2,\dots,n), \tag{2.10}$$

where $\psi(z)$ is single-valued in $\mathbb{D}$ as the derivative of the function with logarithmic multi-valued terms:

$$\psi(z) = \varphi'(z). \tag{2.11}$$

Formula (2.2) yields the Riemann-Hilbert problem

$$\mathrm{I}m\,[(t-a_k)\psi(t)] = 0, \quad t \in L_k\ (k = 1,2,\dots,n) \tag{2.12}$$

on the function $\psi(z)$ analytic in the domain $\mathbb{D}$ and continuous in its closure. Following [9] one can reduce the Riemann-Hilbert problem to the $\mathbb{R}$–linear problem

$$(t-a_k)\,\psi(t) = (t-a_k)\,\psi_k(t) + \overline{(t-a_k)}\,\overline{\psi_k(t)} + \beta_k,\ |t-a_k| = r_k,\ (k = 1,2,\dots,n), \tag{2.13}$$

where ψ_k are analytic in $|z-a_k| < r_k$ and continuous in $|z-a_k| \leqslant r_k$; β_k are undetermined real constants. Further, we shall show that $\beta_k = d_k = A_k$. The problems (2.12) and (2.13) are equivalent in the following sense.

Theorem 1. [9]

(i) If $\psi(z)$ and $\psi_k(z)$ are solutions of (2.13), then $\psi(z)$ satisfies (2.12).

(ii) If $\psi(z)$ is a solution of (2.12), there exist such functions $\psi_k(z)$ and real constants β_k $(k=1,2,\ldots,n)$ that the $\mathbb{R}$–linear condition (2.13) is fulfilled.

The $\mathbb{R}$–linear problem (2.13) can be written in the form

$$\psi(t)=\psi_k(t)+\left(\frac{r_k}{t-a_k}\right)^2\overline{\psi_k(t)}+\frac{\beta_k}{t-a_k},\ |t-a_k|=r_k,\ (k=1,2,\ldots,n). \quad (2.14)$$

Introduce the space $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ of functions analytic in the domain $\mathbb{D}_k=\{z\in\mathbb{C}:|z-a_k|<r_k\}$ and continuous in its closure. This is a Banach space endowed with the norm $||f||=\max_{|t-a_k|=r_k}|f(t)|$. By Maximum Principle convergence in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ means uniform convergence in $\mathbb{D}_k$. The space $\mathcal{C}_{\mathcal{A}}(\mathbb{D})$ is introduce along similar lines.

The $\mathbb{R}$–linear problem (2.14) is reduced to functional equations. Consider the inversion with respect to the circle L_k

$$z_{(k)}^*:=\frac{r_k^2}{z-a_k}+a_k, \quad (k=1,2,\ldots,n) \quad (2.15)$$

Following [9], [10] one can reduce the problem (2.14) to the system of functional equations

$$\psi_k(z)=\sum_{m\neq k}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z_{(m)}^*\right)}+h_k(z), \quad (2.16)$$

where the function

$$h_k(z)=\sum_{m\neq k}\frac{\beta_m}{z-a_m}, \quad |z-a_k|\leqslant r_k,\ k=1,\ldots,n \quad (2.17)$$

belongs to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$. The general solution of the Riemann–Hilbert problem (2.12) is constructed via $\psi_k(z)$ [9], [10]

$$\psi(z)=\sum_{m=1}^{n}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z_{(m)}^*\right)}+\sum_{m=1}^{n}\frac{\beta_m}{z-a_m}, \quad z\in\mathbb{D}\cup\partial\mathbb{D}. \quad (2.18)$$

The function $\psi(z)$ analytic in $\mathbb{D}$.

Theorem 2. ([8, Lemma 4.8, p. 167]) The system (2.16) has a unique solution in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ $(k=1,2,\ldots,n)$. This solution can be found by the method of successive approximations.

Let $\psi_k(z)$ be a solution to the system of functional equations (2.16). Let $w\in\mathbb{D}\backslash\{\infty\}$ be a fixed point. Introduce the functions

$$\varphi_m(z)=\int\limits_{w_{(m)}^*}^{z}\psi_m(t)dt+\varphi_m(w_{(m)}^*), \quad m=1,2,\ldots,n, \quad (2.19)$$

and

$$\omega(z) = -\sum_{m=1}^{n}\left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m\left(w^*_{(m)}\right)}\right] + \sum_{m=1}^{n}\beta_m \ln\frac{z-a_m}{w-a_m}. \tag{2.20}$$

Here, the following relation is used [8]

$$\frac{d}{dz}\left[\overline{\varphi_m\left(z^*_{(m)}\right)}\right] = -\left(\frac{r_k}{z-a_k}\right)^2 \overline{\frac{d\varphi_m}{dz}\left(z^*_{(m)}\right)},\ |z-a_k| > r_k. \tag{2.21}$$

The function $\omega(z)$ differs from $\phi(z)$ introduced in (2.2) by an additive constant. This follows from equations $\omega'(z) = \psi(z)$ and (2.11) where $\psi(z)$ is given by (2.18). The functions $\omega(z)$ and $\varphi_m(z)$ belong to $\mathcal{C}_{\mathcal{A}}(\mathbb{D})$ and to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_m)$, respectively. One can see from (2.19) that the function $\varphi_m(z)$ is determined by $\psi_m(z)$ up to an additive constant which vanishes in (2.20). The function $\omega(z)$ vanishes at $z = w$.

Integrate each functional equation (2.16). Application of (2.19) yields the functional equations with respect to $\varphi_k \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$

$$\varphi_k(z) = -\sum_{m\neq k}\left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m\left(w^*_{(m)}\right)}\right] + \sum_{m\neq k}\beta_m \ln(z-a_m) + C_k, \tag{2.22}$$

$$|z-a_k| \leqslant r_k, \quad k = 1,\dots,n,$$

where C_k are undetermined constants.

Theorem 3. [8] The system (2.22) with fixed C_k has a unique solution in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ ($k = 1,\dots,n$). This solution can be found by the method of successive approximations.

Application of the method of successive approximations to (2.22) yields the uniformly convergent series

$$\varphi_k(z) = \underbrace{C_k + \sum_{k_1\neq k}\beta_{k_1}\ln(z-a_{k_1})}_{\text{0th approx.}} \underbrace{- \sum_{k_1\neq k}\sum_{k_2\neq k_1}\beta_{k_2}\ln\overline{\frac{a_{k_2}-z^*_{(k_1)}}{a_{k_2}-w^*_{(k_1)}}}}_{\text{1st approx.}} \tag{2.23}$$

$$\underbrace{+ \sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}\beta_{k_3}\ln\frac{a_{k_3}-z^*_{(k_2k_1)}}{a_{k_3}-w^*_{(k_2k_1)}}}_{\text{2nd approx.}}$$

$$\underbrace{- \sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}\sum_{k_4\neq k_3}\beta_{k_4}\ln\overline{\frac{a_{k_4}-z^*_{(k_3k_2k_1)}}{a_{k_4}-w^*_{(k_3k_2k_1)}}}}_{\text{3d approx.}} + \dots, \quad |z-a_k| \leqslant r_k.$$

Here, uniform convergence is understood in such a way that each term of the series coincides with a successive approximation shown in (2.23). The order of summation into each approximation can be arbitrarily fixed. It is worth noting that an approximation of order p contains the Möbius mapping of the same level p [7].

The function (2.20) can be written in the form of the uniformly convergent product

$$\omega(z)=\ln\left\{\prod_{k=1}^{n}\left(\frac{a_k-z}{a_k-w}\right)^{\beta_k}\prod_{k_1\neq k}\left(\overline{\frac{a_{k_1}-w^*_{(k)}}{a_{k_1}-z^*_{(k)}}}\right)^{\beta_{k_1}}\times\right. \tag{2.24}$$

$$\prod_{k=1}^{n}\prod_{k_1\neq k}\prod_{k_2\neq k_1}\left(\frac{a_{k_2}-z^*_{(k_1k)}}{a_{k_2}-w^*_{(k_1k)}}\right)^{\beta_{k_2}}\times$$

$$\left.\prod_{k=1}^{n}\prod_{k_1\neq k}\prod_{k_2\neq k_1}\prod_{k_3\neq k_2}\left(\overline{\frac{a_{k_3}-w^*_{(k_2k_1k)}}{a_{k_3}-z^*_{(k_2k_1k)}}}\right)^{\beta_{k_3}}\cdots\right\}.$$

One can see that the milti-valued part of $\omega(z)$ coincides with $\sum_{k=1}^{n}\beta_k\ln(z-a_k)$. The function $\varphi(z)=\omega(z)+C_0$ from (2.2) is determined up to an arbitrary additive constant C_0. Hence, the milti-valued part of $\omega(z)$ and $\varphi(z)$ are the same. This yields $\beta_k=A_k=d_k$ $(k=1,2,\ldots,n)$. Therefore, the function $U(z)$ is exactly written by the formulas $U(z)=\mathrm{Re}\,\omega(z)+\mathrm{Re}\,C_0$ and (2.24), and depends on the additive real constant $\mathrm{Re}\,C_0$. These formulae can be directly applied to the study of the energy functional of the Ginzburg-Landau theory.

Direct application of the formula (2.24) for densely packed holes $\mathbb{D}_k$ gives a slowly convergent product. In this case, one can apply asymptotic analysis of the function $\psi(z)=\omega'(z)$ following [13] and the fast method described in [12].

It is interesting to study zeros of $\psi(z)=\omega'(z)$ to investigate the location of zeros of the Riemann θ-function on the corresponding Schottky double [11], [2]. The zeros of $\psi(z)$ also solve the Jacobi inversion problem. In the previous works, it was shown that the μ-part of (45) (see paper [9]) has zeros only on the boundary $\partial\mathbb{D}$ as the derivative of a conformal mapping [2]. The question, whether the β-part of (45), the function $\psi(z)=\omega'(z)$, has zeros only on $\partial\mathbb{D}$ stays open.

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