Functional Equations for Analytic Functions and Their Application to Elastic Composites

Piotr Drygaś and Vladimir Mityushev

Abstract Two-dimensional elastic composites with non-overlapping inclusions is studied by means of the boundary value problems for analytic functions following Muskhelishvili's approach. We develop a method of functional equations to reduce this problem for a circular multiply connected domain to functional-differential equations. Analytical formulae for the effective constants are deduced.

Keywords Eisenstein series • Functional equation • Natanzon series • Two-dimensional elastic composite

Mathematics Subject Classification (2010) Primary 30E25; Secondary 74Q15

1 Introduction

Two-dimensional elastic composites with non-overlapping inclusions can be discussed through boundary value problems for analytic functions following Muskhelishvili's approach [16]. A method of functional equations was proposed to solve the Riemann–Hilbert and $\mathbb{R}$-linear problems for multiply connected domains [13]. These results were applied to description of the local fields and the effective conductivity tensor for 2D composites [1, 4, 8, 9, 12, 14, 15]. In the present note, we develop this method of functional equations to elastic problems modelled by the biharmonic equation. We reduce the problem for a circular multiply connected domain to a system functional-differential equations and propose a constructive method for their solution in terms of the generalized Eisenstein and Natanzon functions [2, 6, 7, 17]).

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P. Dang et al. (eds.), *New Trends in Analysis and Interdisciplinary Applications*, Trends in Mathematics, DOI 10.1007/978-3-319-48812-7_4

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2 Statement of the Elastic Problem

Consider n disks $D_k = \{z \in \mathbb{C} : |z - a_k| < r\}$, $(k = 1, .., n)$ in the complex plane $\mathbb{C}$. Let $\Gamma_k = \partial D_k$, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\dot{D} = \mathbb{C} \setminus \bigcup_{j=1}^{n} \left(D_j \cup \Gamma_j\right)$, $D = \dot{D} \cup \{\infty\}$ where the circle Γ_k is orientated in counter-clockwise sense. Further, the limit case as $n \to \infty$ will be considered following the method described in [5]. This means that we formally introduce an infinite sequence of non-overlapping disks D_k $(k = 1, 2, \ldots)$. After, we fix a number n and consider only first n disks. Since the number n is arbitrary in our study, hence we can take the limit $n \to \infty$ in the final formulae.

The component of the stress tensor can be determined by the Kolosov–Muskhelishvili formulas [16]

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\mathrm{Re}\varphi_k'(z), \; z \in D_k, \\ 4\mathrm{Re}\varphi_0'(z), \;\; z \in D, \end{cases} \tag{1}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = \begin{cases} -2\left[z\overline{\varphi_k''(z)} + \overline{\psi_k'(z)}\right], \; z \in D_k, \\ -2\left[z\overline{\varphi_0''(z)} + \overline{\psi_0'(z)}\right], \;\; z \in D, \end{cases}$$

Introduce constants $B_0 = \frac{\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty}}{4}$, $\Gamma_0 = \frac{\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty} + 2i\sigma_{xy}^{\infty}}{2}$, where σ_{xx}^{∞}, σ_{xy}^{∞}, σ_{yy}^{∞} are the given stresses at infinity. Introduce the functions $\varphi_0(z) = B_0 z + \varphi(z)$, $\psi_0(z) = \Gamma_0 z + \psi(z)$ where $\varphi(z)$ and $\psi(z)$ are analytical in D and bounded at infinity, $\varphi_k(z)$ and $\psi_k(z)$ are analytical in D_k and all one twice differentiable in the closures of the considered domains. The ideal contact between different materials is expressed by means of the following boundary conditions [16]

$$\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} = \varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}, \tag{2}$$

$$\mu\left(\kappa_1\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)}\right) = \mu_1\left(\kappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}\right), \tag{3}$$

$t \in \partial D_k$. In these equations μ is a shear modulus, κ is Kolosov constant and $\kappa = 3 - 4\nu$ in plane strain; $\kappa = (3 - \nu)/(1 + \nu)$ in plane stress; ν is Poisson's ratio. Index 1 denotes physical constants for inclusions.

3 Functional-Differential Equations

Let $z_{(k)}^{*} = \frac{r^2}{\overline{z - a_k}} + a_k$ denote the inversion with respect to the circle Γ_k. Introduce the functions $\Phi_k(z) = \overline{z_{(k)}^{*}}\varphi_k'(z) + \psi_k(z)$, $|z - a_k| \le r$, analytic in D_k except point a_k, where its principal part has the form $r^2 (z - a_k)^{-1} \varphi_k'(a_k)$. The problem (2), (3) can

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be reduced to the system of functional equations following [11, 13]

$$\left(\frac{\mu_1}{\mu}+\kappa_1\right)\varphi_k(z)=\left(\frac{\mu_1}{\mu}-1\right)\sum_{m\neq k}\left[\overline{\Phi_m(z^*_{(m)})}-(z-a_m)\,\overline{\varphi_m'(a_m)}\right]$$
$$-\left(\frac{\mu_1}{\mu}-1\right)\overline{\varphi_k'(a_k)}\,(z-a_k)$$
$$+\frac{\mu_1}{\mu}(1+\kappa)B_0z+p_0,\ \ |z-a_k|\leq r, \tag{4}$$

and

$$\left(\kappa\frac{\mu_1}{\mu}+1\right)\Phi_k(z)=\left(\kappa\frac{\mu_1}{\mu}-\kappa_1\right)\sum_{m\neq k}\overline{\varphi_m(z^*_{(m)})}$$
$$+\left(\frac{\mu_1}{\mu}-1\right)\sum_{m\neq k}\left(\frac{r^2}{z-a_k}+\overline{a_k}-\frac{r^2}{z-a_m}+\overline{a_m}\right)$$
$$\left[\left(\overline{\Phi_m(z^*_{(m)})}\right)'-\overline{\varphi_m'(a_m)}\right]$$
$$+\frac{\mu_1}{\mu}(1+\kappa)B_0\left(\frac{r^2}{z-a_k}+\overline{a_k}\right)$$
$$+\frac{\mu_1}{\mu}(1+\kappa)\Gamma_0z+\omega(z),\ \ |z-a_k|\leq r,\ \ k=1,2,\ldots,n \tag{5}$$

where

$$\omega(z)=\sum_{k=1}^{n}\frac{r^2q_k}{z-a_k}+q_0, \tag{6}$$

q_0 is a constant and

$$q_k=\varphi_k'(a_k)\left((\kappa-1)\frac{\mu_1}{\mu}-(\kappa_1-1)\right)-\overline{\varphi_k'(a_k)}\left(\frac{\mu_1}{\mu}-1\right),\ \ k=1,2,\ldots,n. \tag{7}$$

The unknown functions $\varphi_k(z)$ and $\Phi_k(z)$ ($k=1,2,\ldots,n$) are related by $2n$ Eqs. (4) and (5).

Introduce the Banach space $\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^{n}D_k\right)$ as the space of functions f of the form $f(z)=f_k(z)$, $z\in D_k$, analytic in $\bigcup_{j=1}^{n}D_k$, endowed with the norm

$$\|f\|^2_{\mathcal{H}^{(2,2)}}:=\sum_{j=1}^{n}\left(\sup_{0<r<r_k}\int_0^{2\pi}|f_j\left(re^{i\theta}+a_j\right)|^2d\theta+\right.$$
$$\left.\sup_{0<r<r_k}\int_0^{2\pi}|f_j'\left(re^{i\theta}+a_j\right)|^2d\theta+\sup_{0<r<r_k}\int_0^{2\pi}|f_j''\left(re^{i\theta}+a_j\right)|^2d\theta\right).$$

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The functional equations contain compositions of $\varphi_k(z)$ and $\Phi_k(z)$ with inversions which define compact operators in the Banach space $\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^{n} D_k\right)$. Hence, the functional equations (4) and (5) can be effectively solved by use of the symbolic computations. After their solution $\varphi(z)$ and $\psi(z)$ can be found

$$\frac{\mu_1}{\mu}(1+\kappa)\,\varphi(z) = \left(\frac{\mu_1}{\mu}-1\right)\sum_{m=1}^{n}\left[\overline{\Phi_m(z^*_{(m)})} - (z-a_m)\,\overline{\varphi_k'(a_k)}\right] + p_0,\ z\in D, \tag{8}$$

$$\frac{\mu_1}{\mu}(1+\kappa)\,\psi(z) = \omega(z) - \left(\frac{\mu_1}{\mu}-1\right)\sum_{m=1}^{n}\left(\frac{r^2}{z-a_m}+\overline{a_m}\right)\left[\left(\overline{\Phi_m(z^*_{(m)})}\right)' - \overline{\varphi_m'(a_m)}\right]$$
$$+\left(\kappa\frac{\mu_1}{\mu}-\kappa_1\right)\sum_{m=1}^{n}\overline{\varphi_m(z^*_{(m)})},\ z\in D. \tag{9}$$

Theorem 3.1 ([3]) *For sufficiently small coefficients μ, μ_k, κ and κ_k $(k=1,\ldots,n)$ the method of successive approximations applied to (4) and (5) converges in $\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^{n} D_k\right)\times\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^{n} D_k\right)$.*

For instance, the zero approximation has the form

$$\varphi_k^{(0)}(z) = -\frac{1-\frac{\mu}{\mu_k}}{1+\frac{\mu}{\mu_k}\kappa_k}\frac{(\kappa+1)}{2-\frac{\mu}{\mu_k}+\frac{\mu}{\mu_k}\kappa_k}B_0\,(z-a_k) + (\kappa+1)B_0 z + p_0, \tag{10}$$

$$\psi_k^{(0)}(z) = (1+\kappa)\Gamma_0 z + q_0,\ |z-a_k|\le r,\ k=1,2,\ldots,n. \tag{11}$$

Let the approximation of the order $(p-1)$ be known. Then the p-th approximation for $\varphi_k^{(p)}(z)$ has the form

$$\varphi_k^{(p)}(z) = \left(1+\frac{\mu}{\mu_k}\kappa_k\right)^{-1}$$
$$\times\left(\sum_{m\neq k}\left(1-\frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m^{(p-1)}(z^*_{(m)})} - (z-a_m)\overline{\left(\varphi_m^{(p-1)}\right)'(a_m)}\right]\right.$$
$$\left. - \frac{\left(1-\frac{\mu}{\mu_k}\right)(\kappa+1)B_0}{2-\frac{\mu}{\mu_k}+\frac{\mu}{\mu_k}\kappa_k}(z-a_k) + (\kappa+1)B_0 z + p_0\right). \tag{12}$$

Analogous formulae can be written for $\psi_k^{(p)}(z)$.

An alternative method is to solve Eqs. (4) and (5) by series with undetermined coefficients proposed in [11]. We are looking for the analytic potentials φ_k and ψ_k

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in the form of the series in r^2

$$\varphi_k(z) = \sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\alpha_{k,j}^{(s)} r^{2s}(z-a_k)^j,\ \ \psi_k(z) = \sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\beta_{k,j}^{(s)} r^{2s}(z-a_k)^j. \tag{13}$$

Selecting the terms with the same powers $(z - a_k)^j$ and r^{2s} we arrive at an iterative method to find $\alpha_{k,j}^{(p)}$ and $\beta_{k,j}^{(p)}$. When the coefficients are determined, the functions (13), hence (8) and (9), can be approximately constructed. The stress and deformation tensors can be calculated by the Kolosov–Muskhelishvili formulae [16]. Then, the effective elastic moduli of macroscopically isotropic fibrous composites can be calculated as follows:

$$\mu_{\text{eff}} = \frac{\langle \sigma_{xx} - \sigma_{yy}\rangle}{2\langle \varepsilon_{xx} - \varepsilon_{yy}\rangle},\ k_{\text{eff}} = \frac{\langle \sigma_{xx} + \sigma_{yy}\rangle}{2\langle \varepsilon_{xx} + \varepsilon_{yy}\rangle}. \tag{14}$$

Here, the limit average over the plane is introduced

$$\langle A\rangle = \lim_{Q_n\to\infty}\frac{1}{|Q_n|}\iint_{Q_n} A\, dxdy.$$

In the latter limit, it is assumed that the infinitely many points a_k are distributed in the plane. After long symbolic computations we get

$$\begin{aligned}\mu_{\text{eff}} = \mu &- \frac{(\kappa+1)\mu\,(\mu-\mu_1)}{\kappa\mu_1+\mu} f \\ &+ \Bigg(-\frac{2B_0(\kappa+1)\mu(\mu-\mu_1)(-\kappa\mu_1+(\kappa_1-1)\mu+\mu_1)}{\pi\Gamma_0(\kappa\mu_1+\mu)((\kappa_1-1)\mu+2\mu_1)} \\ &+ \frac{2e_3^{(1)}(\kappa+1)\mu\overline{\Gamma_0}(\mu-\mu_1)^2}{\pi\Gamma_0(\kappa\mu_1+\mu)^2} + \frac{\kappa(\kappa+1)\mu(\mu-\mu_1)^2}{(\kappa\mu_1+\mu)^2}\Bigg) f^2 + O(f^3)\end{aligned} \tag{15}$$

and

$$\begin{aligned}k_{\text{eff}} = 7k &+ \frac{k(\kappa+1)\mu\,(-\kappa_1\mu+(\kappa-1)\mu_1+\mu)}{(\kappa-1)\,((\kappa_1-1)\,\mu+2\mu_1)} f \\ &+ \Bigg(\frac{e_2^{(0)}k(\kappa+1)\,(\mu-\mu_1)\,\overline{\Gamma_0}\,(\kappa_1\mu+\mu_1)\,(-\kappa_1\mu+\kappa\mu_1+\mu-\mu_1)}{\pi B_0(\kappa-1)\,(\kappa_1+1)\,(\kappa_1\mu-\mu+2\mu_1)\,(\kappa\mu_1+\mu)} \\ &\quad + \frac{\overline{e_2^{(0)}}k\Gamma_0(\kappa+1)\,(\mu-\mu_1)^2\,(-\kappa_1\mu+(\kappa-1)\mu_1+\mu)}{\pi B_0(\kappa-1)\,(\kappa_1+1)\,((\kappa_1-1)\,\mu+2\mu_1)\,(\kappa\mu_1+\mu)}\end{aligned}$$

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$$
- \frac{2k(\kappa+1)\mu\left(-\kappa_1\mu+(\kappa-1)\mu_1+\mu\right)\left((\kappa_1-1)\mu-\kappa\mu_1+\mu_1\right)}{(\kappa-1)^2\left((\kappa_1-1)\mu+2\mu_1\right)^2}\Bigg) f^2
$$

$$
+O(f^3), \tag{16}
$$

where

$$
e_2^{(0)} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sum_{m\neq k} \frac{1}{(a_k-a_m)^2}, \quad e_3^{(1)} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sum_{m\neq k} \frac{\overline{a_k-a_m}}{(a_k-a_m)^3} \tag{17}
$$

are the generalized Eisenstein and Natanzon series [2, 3, 10, 17] which depend on the values of B_0 and Γ_0. The Eisenstein summation must be applied for the conditionally convergent sums (17). Take $B_0 = 0$, $\Gamma_0 = i$ in (15) and $B_0 = 1$, $\Gamma_0 = 0$ in (16). Then, we get $e_2^{(0)} = \pi$ (see discussion in [10, 15]). It is a generalization of the Rayleigh formula [18] for a regular array. Numerical simulations suggest that $e_3^{(1)} = \frac{\pi}{2}$. A rigorous proof of the latter formula has been unknown.

Acknowledgements Piotr Drygaś was partially supported by the Centre for Innovation and Transfer of Natural Science and Engineering Knowledge of University of Rzeszow (grant No. WMP/GD-04/2015).

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