Conformal Mapping of Circular Multiply Connected Domains Onto Domains with Slits

Roman Czapla and Vladimir V. Mityushev

Abstract The conformal mapping of the square with circular disjoint holes onto the square with disjoint slits is constructed. This conformal mapping is considered as a solution of the Riemann–Hilbert problem for a multiply connected domain in a class of double periodic functions. The problem is solved by reduction to a system of functional equations.

Keywords Circular multiply connected domain • Conformal mapping • Multiply connected domain with slits • Riemann–Hilbert problem

1 Introduction

Analytical formulae for conformal mapping of multiply connected domains with slits onto circular domains are the canonical formulae of complex analysis. Such a formula can be referred to the Schwarz–Christoffel formula. Analytical formulae for conformal mapping between various canonical slit domains with sufficiently well-separated boundary components were given in [3] and the works cited therein. The geometrical restrictions were eliminated in [6] and a formula for an arbitrary circular multiply connected domain was constructed by means of the Poincaré series. It is worth noting that the Poincaré series is the second derivative of the Schottky–Klein prime function [3] that makes a possibility to use various form of analytical formulae. The uniform convergence of the Poincaré series for an arbitrary circular multiply connected domain can extend the validity of the constructions by DeLillo et al [3] by modifications explained in [6].

Analogous study was performed in [7] where the Schwarz–Christoffel formula for conformal mapping of multiply connected domains bounded by polygons onto circular domains was constructed. The formula given in [7] does not contain any geometrical restriction on boundary components. However, the general Schwarz–Christoffel formula contains the accessory parameters contrary to formulae for

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conformal mappings onto slit domains. This is the reason why the special formula [6] for slit domains is preferable than the general one.

Besides the canonical slit domains, conformal mapping of multiply connected domains with slits of various inclinations onto circular domains is applied to boundary value problems of fracture mechanics. The results of [6] were developed to domains bounded by mutually disjoint arbitrarily oriented slits in [2].

The above presented results concern the canonical multiply connected domains bounded by disjoint circles. However, a straight line can be treated a circle on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$. This fact was used in [4] to extend the method of functional equations [10] to strips and rectangles with circular holes.

In this paper, we follow the method outlined in [4] to construct the conformal mapping of the square with circular holes onto the square with slits of given inclinations. The conformal mapping is constructed as a solution of the Riemann–Hilbert problem for a doubly periodic multiply connected domain. The latter problem is reduced to a system of functional equations.

2 Basic Problem

Let $z = x + iy$ denote a complex variable on the complex plane $\mathbb{C}$ and $\mathcal{G} = \left\{z \in \mathbb{C}\colon \left|\mathrm{Re}[z]\right| \le \frac{1}{2} \ \& \ \left|\mathrm{Im}[z]\right| \le \frac{1}{2}\right\}$ stands for the unit square. Consider non-overlapping disks $D_k = \{z \in \mathcal{G}\colon |z - a_k| < r_k\}$ $(k = 1, 2, \dots, N)$. Let $\mathcal{D}$ denote the complement of the closed disks $|z - a_k| \le r_k$ to the square $\mathcal{G}$. Consider the second complex variable $\zeta = u + iv$ on the complex plane with non-overlapping slits Γ_k lying in the square $\mathcal{G}$. Here, each slit Γ_k has two sides, hence, it is considered as a closed curve. Let $\mathcal{D}'$ stand for the complement of all the slits Γ_k to $\mathcal{G}$. Find a conformal mapping φ of the circular multiply connected domain $\mathcal{D}$ onto $\mathcal{D}'$. Function φ has to satisfy the following boundary conditions:

$$\mathrm{Im}\left[e^{-i\alpha_k}\varphi(t)\right] = c_k, \quad |t - a_k| = r_k,\ k = 1, 2, \dots, N \tag{1}$$

$$\mathrm{Re}\left[\varphi\left(\pm\frac{1}{2} + iy\right)\right] = \pm\frac{1}{2}, \quad -\frac{1}{2} \le y \le \frac{1}{2}, \tag{2}$$

$$\mathrm{Im}\left[\varphi\left(x \pm \frac{i}{2}\right)\right] = \pm\frac{1}{2}, \quad -\frac{1}{2} \le y \le \frac{1}{2}, \tag{3}$$

where α_k is the given inclination angle of Γ_k, a_k and r_k are center and radius of the k-th disk, c_k are undetermined real constants. The condition (1) means that each slit Γ_k lies on the line $-\sin\alpha_k u + \cos\alpha_k v = c_k$, i.e., the circle $|z - a_k| = r_k$ maps onto the slit Γ_k. The conditions (2) and (3) show that the boundary of the square $\mathcal{G}$ is mapped onto itself and four corner points are fixed points of the conformal mapping. The fixed point condition should not be imposed in general case. But we shall consider only symmetric domains $\mathcal{D}$ and $\mathcal{D}'$ when this condition is automatically fulfilled.

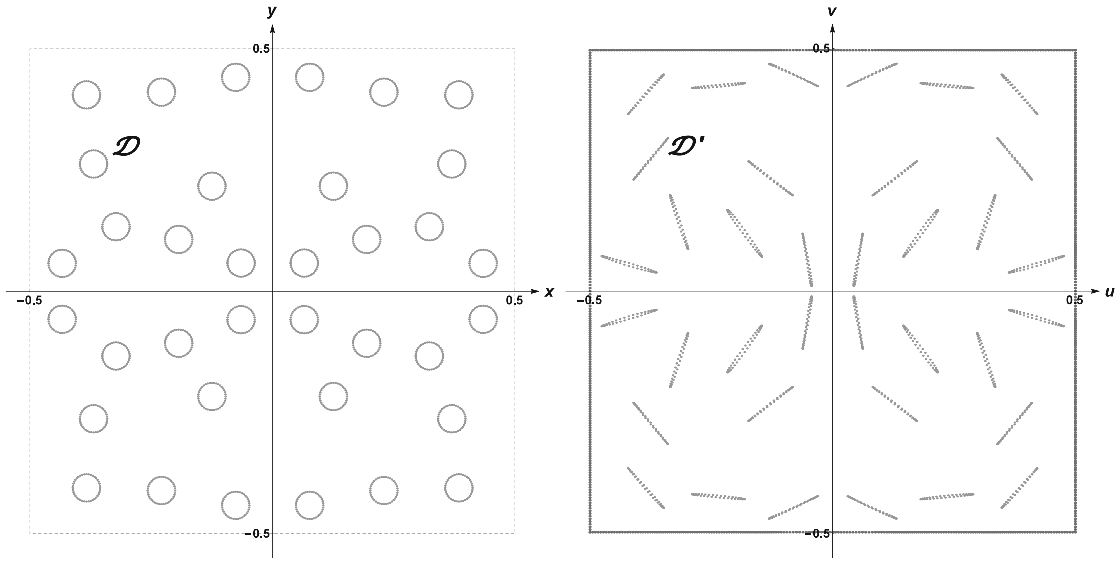

Fig. 1 Domains $\mathcal{D}$ and $\mathcal{D}'$

More precisely, we assume that the holes of $\mathcal{D}$ and the slits of $\mathcal{D}'$ are symmetric with respect to the axes as shown in Fig. 1 that yields the symmetry of the conformal mapping including the symmetry of the constants c_k and $\varphi(0) = 0$. Instead of the symmetric square one can consider the one fourth small square without any symmetry condition.

The conditions (1)–(3) can be considered as the Riemann–Hilbert problem which has a unique solution if one of the undetermined constants c_k is fixed [10]. Therefore, the conformal mapping and the unique solution of the Riemann–Hilbert problem (1)–(3) coincide. This implies the uniqueness of the discussed conformal mapping. In the same time, this implies that the unique solution of (1)–(3) is a univalent function.

We now proceed to solve the problem (1)–(3). Introduce an auxiliary function $\widetilde{\varphi}(z) = \varphi(z) - z$. Then (1)–(3) become

$$\operatorname{Im}\left[e^{-i\alpha_k}\widetilde{\varphi}(t)\right] = c_k - \operatorname{Im}\left[e^{-i\alpha_k}t\right], \quad |t - a_k| = r_k, \ k = 1, 2, \ldots, N, \tag{4}$$

$$\operatorname{Re}\left[\widetilde{\varphi}\left(\pm\frac{1}{2} + iy\right)\right] = 0, \quad -\frac{1}{2} \leq y \leq \frac{1}{2}, \tag{5}$$

$$\operatorname{Im}\left[\widetilde{\varphi}\left(x \pm \frac{i}{2}\right)\right] = 0, \quad -\frac{1}{2} \leq x \leq \frac{1}{2}. \tag{6}$$

The problem (4) can be reduced to $\mathbb{R}$-linear problem [10]

$$\widetilde{\varphi}(t) = \varphi_k(t) + e^{2i\alpha_k}\overline{\varphi_k(t)} + ie^{i\alpha_k}c_k - t, \qquad |t - a_k| = r_k, \tag{7}$$

where φ_k are analytic in $|z - a_k| < r_k$ and continuously differentiable in $|z - a_k| \leq r_k$, $k = 1, 2, \ldots, N$. The equivalence of the boundary conditions (4) and (7) was justified in [2] and [6]. Multiply Eq. (7) by $e^{-i\alpha_k}$:

$$e^{-i\alpha_k}\widetilde{\varphi}(t) = e^{-i\alpha_k}\varphi_k(t) + e^{i\alpha_k}\overline{\varphi_k(t)} + ic_k - e^{-i\alpha_k}t$$

that is equivalent to the relation

$$e^{-i\alpha_k}\widetilde{\varphi}(t) = 2\mathrm{Re}\left[e^{-i\alpha_k}\varphi_k(t)\right] + ic_k - e^{-i\alpha_k}t. \tag{8}$$

One can see that the imaginary part of (8) yields (4). The inverse way from (4) to (8) is based on the solution to the Dirichlet problem for the disk D_k

$$2\mathrm{Re}\left[e^{-i\alpha_k}\varphi_k(t)\right] = \mathrm{Re}\left[e^{-i\alpha_k}\widetilde{\varphi}(t) + e^{-i\alpha_k}t\right]. \tag{9}$$

with respect to $e^{-i\alpha_k}\varphi_k(z)$ with given $\widetilde{\varphi}$ (for details see [10]).

We now demonstrate that $\widetilde{\varphi}$ is a double periodic function:

$$\widetilde{\varphi}(z+i) = \widetilde{\varphi}(z) = \widetilde{\varphi}(z+1). \tag{10}$$

The symmetry of the conditions (4)–(6) with respect to the x-axis and y-axis implies that

$$\overline{\widetilde{\varphi}(\overline{z})} = \widetilde{\varphi}(z), \quad z \in \mathcal{D} \tag{11}$$

$$\overline{-\widetilde{\varphi}(-\overline{z})} = \widetilde{\varphi}(z). \quad z \in \mathcal{D}; \tag{12}$$

It follows from (11) that $\overline{\widetilde{\varphi}\left(x+\frac{i}{2}\right)} = \widetilde{\varphi}\left(x-\frac{i}{2}\right)$ for $\frac{1}{2} \leq x \leq \frac{1}{2}$, hence $\mathrm{Re}\ \widetilde{\varphi}\left(x+\frac{i}{2}\right) = \mathrm{Re}\ \widetilde{\varphi}\left(x-\frac{i}{2}\right)$. Using (6) we get $\widetilde{\varphi}\left(x+\frac{i}{2}\right) = \widetilde{\varphi}\left(x-\frac{i}{2}\right)$. In the same way, the condition (12) gives (5).

The $\mathbb{R}$-linear problem (7) can be reduced to functional equations. We introduce the function [10]:

$$\Phi(z) = \begin{cases} \varphi_k(z) - \sum\limits_{m=1}^{N} e^{2i\alpha_m} \sum\limits_{m_1,m_2}^{*} \left[\overline{\varphi_m\left(\overline{\frac{r_m^2}{z-a_m-m_1-m_2 i}} + a_m\right)} - \overline{\varphi_m(a_m)}\right] + f_k(z), \\ \qquad |z-a_k| \leq r_k, \qquad k = 1,2,\ldots N, \\ \widetilde{\varphi}(z) - \sum\limits_{m=1}^{N} e^{2i\alpha_m} \sum\limits_{m_1,m_2} \left[\overline{\varphi_m\left(\overline{\frac{r_m^2}{z-a_m-m_1-m_2 i}} + a_m\right)} - \overline{\varphi_m(a_m)}\right], \ z \in \mathcal{D}, \end{cases} \tag{13}$$

where $f_k(z) = ie^{i\alpha_k}c_k - z + e^{2i\alpha_k}\overline{\varphi_k(a_k)}$ and

$$\sum_{m=1}^{N} \sum_{m_1,m2}^{*} V_{(m_1,m_2)m} := \sum_{m\neq k} \sum_{m_1,m_2} V_{(m_1,m_2)m} + \sum_{m_1,m_2}{}' V_{(m_1,m_2)k}.$$

In the sum $\sum'_{m_1,m_2}$ the integer numbers m_1, m_2 range from $-\infty$ to $+\infty$ except the case when $m_1^2 + m_2^2 = 0$.

We calculate the jump across the circle $|t - a_k| = r_k$

$$\Delta_k := \Phi^+(t) - \Phi^-(t), \quad |t - a_k| = r_k,$$

where $\Phi^+(t) := \lim_{z \to t\, x \in \mathcal{D}} \Phi(z)$ and $\Phi^-(t) := \lim_{z \to t\, x \in D_k} \Phi(z)$. Using (7) we get $\Delta_k = 0$.

The definition of Φ in $|z - a_k| \le r_k$ ($k = 1, 2, \ldots, N$) yields the following system of functional equations

$$\varphi_k(z) - \sum_{m=1}^{N} e^{2i\alpha_m} \sum_{m_1,m_2}^{*} \left[\overline{\varphi_m\left(\frac{r_m^2}{\overline{z - a_m - m_1 - m_2 i}} + a_m \right)} - \overline{\varphi_m(a_m)} \right] + f_k(z) = C. \tag{14}$$

Let φ_k ($k = 1, 2, \ldots, N$) be a solution of (14). Then the function $\widetilde{\varphi}$ can be found from the definition of Φ in $\mathcal{D}$

$$\widetilde{\varphi}(z) = \sum_{m=1}^{N} e^{2i\alpha_m} \sum_{m_1,m_2} \left[\overline{\varphi_m\left(\frac{r_m^2}{\overline{z - a_m - m_1 - m_2 i}} + a_m \right)} - \overline{\varphi_m(a_m)} \right] + C. \tag{15}$$

3 Numerical Example

Solution to the functional equations (14) can be found by the method of approximations [1]. The zero order approximation is

$$\varphi_k^{(0)}(z) = C - i e^{i\alpha_k} c_k + z.$$

Then (15) implies that

$$\widetilde{\varphi}^{(0)}(z) = \sum_{m-1}^{N} e^{2i\alpha_m} \left[r_m^2 \sum_{m_1,m_2} \left(\frac{1}{\overline{z - a_m - m_1 - m_2 i}} \right) \right], \tag{16}$$

where C is taken to be zero. The zero order approximation of function φ has the form

$$\varphi^{(0)}(z) = \sum_{m=1}^{N} e^{2i\alpha_m} r_m^2 E_1(z - a_m) + z, \tag{17}$$

where E_1 denotes the Eisenstein function which can be expressed in terms of the Weierstrass ζ-function [11]

$$E_1(z) = \sum_{m_1,m_2} \frac{1}{z - m_1 - im_2} = \zeta(z) - \pi z. \tag{18}$$

One can expect that the zero approximation (17) gives sufficiently good results for not high density of slits. Let $\mathcal{G}$ be the unit square. Consider 36 non-overlapping circular disks D_k of radius $r = 0.028$ symmetric with respect to the x-axis and y-axis distributed in $\mathcal{G}$. Conformal mapping of the considered domain $\mathcal{D}$ onto the square with slits of the inclinations randomly chosen on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is presented in Fig. 1. Higher order approximations will be constructed in a separate paper.

4 Discussion

The successive approximations applied to (14) yield an infinite series after substitution into (15). This is the generalized Poincaré series [8] constructed for doubly periodic generators (inversions with respect to circles composed with translations on the square lattice). The present result demonstrates that the method of functional equations to solve the Riemann–Hilbert problem for complicated multiply connected domains obtained from circular ones by symmetries with respect to straight lines yields generalized Poincaré series, i.e., solves the considered problem exactly. In his plenary ISAAC 2015 talk, Darren Crowdy discussed particular cases of the above problems by use of the Schottky–Klein prime function $S(z)$ constructed for a class of domains restricted by the separation condition and the Fourier–Mellin transforms. One can see that the derivative of $\ln S(z)$ is the Poincaré θ_1-series. This simple observation shows that the most general constructions of solutions to such a type of boundary value problems can be found in [4–10] and the works cited therein.

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