Boundary value problems in of porous media mechanics

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1 Problems in porous media

The major purpose of this report is to present boundary value problems and related questions appearing in the study of transport properties of various mechanical objects and to state new problems having important applications in porous media. We discuss here the simplest transport processes that might occur in a heterogeneous system, namely diffusion of a solute and viscous flow at low Reynolds numbers.

Artificial and natural porous media have a very complex geometric structure, but they are invariant in a wide sense. The simplest translational invariance is modeled by the periodicity of the medium. Invariance may occur for random stationary fields or self-similar sets (fractals). The latter ones can also have a random structure! Numerical generations of such media is given in [1]. Geological media can be considered as complicated fractured networks. Study of their transport properties is an important task of geophysics. Some mathematical problems and methods of their solution are presented in [2].

In general properties of porous media are governed by partial differential equations. Macroscopical properties can be considered as functionals depending on the solution of a boundary value problem for such an equation or sets

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of such equations. Here for clarity, we discuss simple stationary processes which are derived from the classical equations of mathematical physics.

Diffusion processes are governed by the Laplace equation

$$\nabla^2 c = 0, \tag{1}$$

where c is the concentration of a solute. Heat conduction, flow of an electric current, the dielectric constant are also governed by the Laplace equation [1], [18]. The flow of a viscous fluid is governed by the Stokes equation [1]

$$\mu\nabla^2 \mathbf{v} = \nabla p,\ \nabla \cdot \mathbf{v} = 0, \tag{2}$$

where $\mathbf{v}$, p and μ are the velocity, pressure and viscosity of the fluid, respectively. Longitudinal laminar flow between unidirectional cylinders is governed by the two-dimensional Poisson equation [1]

$$\nabla^2 w = 1, \tag{3}$$

where w is the component of velocity which is parallel to cylinders; μ and the pressure gradient are taken equal to 1. In general c and w satisfy a classical boundary condition (Dirichlet, Neumann and their generalizations); $\mathbf{v}$ vanishes on ∂D, the boundary of a domain D, where equations (2) are fulfilled. If D is a Lyapunov's curve (surface) with a finite number of singularities because of angles and sources, then we deal with the classical theory of boundary value problems. But the complex and random geometry of porous media requires the study of boundary value problems with a complicated boundary ∂D. This boundary can be, for instance, a deterministic or a random fractal [1]. Moreover, equations (1) - (2) are also considered on surfaces [2]. The boundary of an infinitely connected domain can be considered as a fractal in the topology of the extended complex plane. Here one can note that the problems with such boundaries can be treated as the boundary value problems with infinite index (winding number) [5]. The natural relation to the entire and meromorphi functions is noted in Sec. 4.

Many theoretical and experimental forces are applied to evaluate the effective properties of the medium. More precisely, let us consider a porous medium of macroscopic dimension L. If we assume that the medium is statistically homogeneous at a length scale $\ell << L$, we can extract a unit cell Q representing the medium. Let $\langle \mathbf{a} \rangle$ denote the average $\langle \mathbf{a} \rangle := \frac{1}{|Q|} \int_{\partial D} \mathbf{R} ds \cdot \mathbf{a}$, where $D \subset Q$, $\mathbf{R}$ denotes position and $|Q|$ the area of Q. The concentration flux with a normalized unit diffusivity can be expressed as $\mathbf{q} := -\nabla c$ in D. Then the effective diffusion tensor $\mathbf{D}$ is defined by the relation

$$\langle \mathbf{q} \rangle = -\mathbf{D} \langle \nabla c \rangle . \tag{4}$$

For simple structures determined by the domain D the tensor $\mathbf{D}$ is correctly defined [6] by a variational problem.

Despite the great success of numerical calculations (see [1], [2] and papers cited therein), many unsolved questions arise if geometrical structure of the medium is complicated. Some theoretical questions are discussed in the next section.

2 Boundary value problems appearing in porous media and methods of complex analysis in the complex plane

Methods of complex analysis constitute one of the most powerful tools of continuum mechanics. They allows us to get constructive results mainly in two-dimensional problems.

Any function $c(x, y)$ satisfying the Laplace equation (1) in a simply connected domain D is represented as the real part of a function analytic in D on $z = x + iy$. This analytic function is called the complex potential. Any function $w(x, y)$ satisfying the Poisson equation (3) is decomposed as follows $w = u + w_0$, where u is harmonic and w_0 is a partial solution of (3). Hence, using complex potentials one can express w in terms of an analytic function. The Dirichlet problem $u = f$ on ∂D for (1) becomes

$$\operatorname{Re} \varphi(z) = f(t), \ t \in \partial D, \tag{5}$$

where $\varphi(z) = u(z) + iv(z)$ is a complex potential. The Neumann problem $\frac{\partial u}{\partial n} = g$ on ∂D after application the Cauchy - Riemann equation $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s}$ and integration on natural parameter s becomes

$$\operatorname{Im} \varphi(z) = f(t), \ t \in \partial D, \tag{6}$$

where f is a primitive function for g.

Many problems of porous media and mechanics of composite materials have the form of the conjugation condition

$$u^{+}=u^{-},\ \frac{\partial u^{+}}{\partial n}=\lambda_k\frac{\partial u^{-}}{\partial n}\ \text{on}\ \partial D_k,\ k=1,2,...,n \tag{7}$$

with respect to the function $u(x,y)$ sectionally harmonic in D^{+} and D^{-}. Here $D^{+}\cup clD^{-}$ generates a simply connected domain G, $D^{-}:=\cup_{k=1}^{n}D_k$, clD^{-} denotes the closure of the domain D^{-}; D_k are simply connected domains modeling inclusions of conductivity (diffusivity) λ_k in the host material D^{+} of unit conductivity. Condition (7) corresponds to perfect contact between different materials. The Dirichlet and Neumann conditions might be given on the exterior boundary ∂G. Condition (7) can be written as the $\mathbb{R}-$linear conjugation condition

$$\varphi^{+}(t)=\varphi^{-}(t)-\rho_k\overline{\varphi^{-}(t)},\ t\in\partial D_k,\ k=1,2,...,n. \tag{8}$$

Here $\rho_k:=(\lambda_k-1)(\lambda_k+1)^{-1}$. If $\lambda_k\to+\infty$, i.e., the conductivity of the inclusion is large in comparison with the matrix conductivity, (8) becomes (5). If $\lambda_k=0$, i.e., D_k is insulating, (8) becomes (6). Conversely, (6) can be written as

$$\varphi^{+}(t)=\varphi^{-}(t)-\rho(t)\overline{\varphi^{-}(t)}+f(t),\ t\in\partial D, \tag{9}$$

where $\rho(t)=-1$, $\varphi^{-}(z)$ is an appropriate function analytic in D^{-}. See for details [12, 16].

Let ∂D be divided into two parts L_1 and L_2. If (5) is given in the interior of L_1, if (6) is given in the interior of L_2, and if φ is bounded at the boundary points of L_1 and L_2, we arrive at the Riemann - Hilbert problem

$$\operatorname{Re}\lambda(t)\varphi(z)=f(t),\ t\in\partial D, \tag{10}$$

where $\lambda(t)=1$ on L_1 and $\lambda(t)=-i$ on L_2. Solution to problem (10) for a simply connected domain is given in the classical books [4, 8]. If D is an $n-$connected domain, the complex potential becomes

$$\varphi(z)+\sum_{k=1}^{n}A_k\ln(z-z_k), \tag{11}$$

where $\varphi(z)$ is a function analytic and single-valued in D; z_k are fixed points from D_k; A_k are real constants which should be determined. A method of integral equations for multiply connected domains was applied in [4, 7, 8]. The problem (10) for multiply connected domain has been solved in [13, 16] in closed form by the method of functional equations.

Analytic functions are also used to study the two-dimensional Stokes equation (2). The following Kolosov - Muskhelishvili formula $\mu\mathbf{v} = \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)},\ z = x + iy \in D$, expresses the velocity $\mathbf{v}$ in terms of complex potentials $\varphi(z)$and $\psi(z)$ [7, 9]. The no-slip condition becomes

$$\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = f(t),\ t \in \partial D, \tag{12}$$

where the known Hölder continuous function f models external field.

3 Boundary value problems appearing in porous media and methods of complex analysis on the torus

The conception of the unit cell derived in Sec.1 leads in a natural way to boundary value problems of the analytic function theory on the torus $\mathcal{Q}$ which appears by identification of the opposite sides of the unit cell Q. For a multiply connected domain on $\mathcal{Q}$, the complex potential (11) becomes

$$\varphi(z) + \sum_{k=1}^{n} A_k \left[\ln\sigma(z - z_k) + z_k\zeta(z - z_k)\right], \tag{13}$$

where σ and ζ are the Weierestrass functions, $\sum_{k=1}^{n} A_k = 0$. Therefore, we obtain the problems (5) - (10), (7) - (12) on the torus $\mathcal{Q}$.

The most studied problem (9) on $\mathcal{Q}$ is the case of the circle $\partial D = \{t \in \mathbb{C} :\ |t| = r\}$. The Laplace and the Poisson equations for the circle are solved in [14] and in [10], respectively by the method of functional equations. This method is extended in [11] to the case of many inclusions on the torus $\mathcal{Q}$. Analytical formulae for the effective permeability have been deduced. Such formulae are very useful in applications because they contain the concentration of inclusions and their positions in symbolic form. Note that the symbolic computations have been performed with *Mathematica*. For instance, for the simple square array of cylinders, we obtain

$$k_c^* = \ln\phi^{-1} - 1.47644 + 2\phi - 0.5\phi^2 - 0.0509713\phi^4 + 0.077465\phi^8 - \tag{14}$$

$0.109757\phi^{12}+0.122794\phi^{16}-0.146135\phi^{20}+0.244536\phi^{24}-0.322667\phi^{28}+O(\phi^{32})$,

where $\phi = \pi r^2$ is the area fraction of the cylinders. Moreover, potentials in analytic form can be used as a first approximation to solve more complicated partial differential equations with the help of numerical methods and computers.

4 Discussion of problems with complex geometry

A random porous medium is characterized by the random positions and the random geometry of the inclusions. The boundary condition (8) holds along random contours ∂D_k ($k = 1, 2, ..., n$). In the paper [11] it is assumed that each contour ∂D_k is a circle, which is characterized by two random parameters, the radius r_k and the centre a_k of each inclusion. The expected value of the effective permeability is calculated in the simple case where the centers a_k are uniformly distributed in the unit cell. In [11], double integrals of the following form were calculated

$$\int \int_Q F(z-w) ds_z ds_w, \tag{15}$$

where $F(z)$ is an elliptic function, for instance, $\ln \lfloor \sigma(z) \rfloor$, $\zeta(z)$, or the Eisenstein functions $E_l(z)$ $(l = 2, 3, ...)$ [19, 10]. One of the formulae from [11] for a rectangular unit cell of the dimension $\alpha \times \alpha^{-1}$ is $\int_Q \mathcal{P}(z-w) ds_w = \pi - S_2$, where $S_2 = 2\zeta(\alpha/2)$ is the well known Rayleigh sum of second order. If the centres a_k are arbitrarily located on the complex plane, a problem of construction of meromorphic functions with prescribed singularities arises. Examples of such functions are constructed in [15] by analogy with the construction of elliptic functions.

Problems of porous media on fractal sets are discussed in [1], [2]. There are two types of problems. The first type appears, when we consider a porous medium with many inclusions arranged in a multiscale structure and when we try to evaluate the influence of large and small inclusions on the macroscopic properties, when the number of small inclusions is large and the number of large inclusions is small. This problem can be treated as the problem of the behavior of the ensemble of particles. Can the ensemble be replaced by a single large element? This question has been partially solved in [11] by reducing the problem to the problem of approximating of Riemannian sums by Riemannian integrals. For doubly periodic media the elliptic functions are integrands in these integrals. In the general case, the problem can be considered as the problem of substituting the system of functional equations

$$\psi_k(z) = \rho \sum_{m \neq k} \left(\frac{r}{z-a_m} \right)^2 \overline{\psi_k \left(\frac{r^2}{\overline{z-a_m}} + a_m \right)} + g_k(z),\ |z-a_k| \leq r,\ k = 1, 2, ..., n. \tag{16}$$

by the integral equation

$$\Psi(w) = \frac{\rho v}{\pi} \int\int_F \frac{\overline{\Psi(z)}}{(z-w)^2} d\sigma_z + G(w),\ w \in F. \tag{17}$$

Here, v is the area fraction of the circular inclusions $|z-a_k| \leq r$ in the unit cell Q, $d\sigma_z = dxdy$. It follows from the physical point of view that (16) can be replaced by (17) for inclusions diluted in $F \subset Q$. If v is not small, (16) should yield a more complicated integral equation. Moreover, there is an interesting relation between (17) and the $\mathbb{R}-$linear conjugation problem [3]. If we deal with a fractal, then the structure of the measure $d\sigma_z$ can be very complex.

The second type of problems appears, when flow in a channel with fractal boundaries is studied. It is worth noting that the classical theory of potentials does not work with fractal curves. Ponomarev [17] has constructed a Borel measure μ on the Koch curve L so that the Cauchy integral $\Phi(z) = \frac{1}{2\pi i} \int_L (t-z)^{-1} d\mu(t)$ is continuous in $\mathbb{C} \cup \{\infty\}$, analytic in $\mathbb{C} \cup \{\infty\} \setminus L$, but $\Phi(z)$ is not constant.

Fractured media can also be considered as particular porous media. In this case boundary value problems on surfaces appear [2]. Boundary value problems on surfaces and for anisotropic media are equivalent. Hence, the complex variable technique [18] can be successfully applied to problems on surfaces. When fractured porous media are modeled by minimal surfaces, it is known that there exists a conformal structure on each minimal surface. Hence, a boundary value problem appears in a Riemann surface [20] in a natural way. The problem is to find this conformal structure. Therefore, the derivation of a minimal surface as a Riemann surface with global uniformisation is a constructive method to solve boundary value problems on surfaces. Some simple examples of such approaches can be found in [16].

In general, fractured media are modeled by a network composed of many surfaces [2]. Hence, boundary value problems arise in on configurations consisting of surfaces. See examples in [16].

5 Conclusion

We briefly presented some general approaches that we have extensively developed and that are related to boundary value problems of the analytic functions theory. Many of the problems are only stated and their solution even in particular cases will be useful for applications in porous media.

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