On the construction of Abelian differentials on closed Riemannian surface

Vladimir Mityushev

1.Introduction.

A differential holomorphic anywhere on the closed Riemannian surface is called an Abelian differential of the first kind. The Abelian differentials generate a vector space . The complex dimension of the space is equal to the genus of the surface. The Abelian differentials has been discussed in [1,2]. In particular it is known the theorem of existence. But the explicit form of the Abelian differentials is known only in some particular cases [1-6]. The Riemann boundary value problem on closed Riemannian surfaces has been solved in [5] in terms of the Abelian differentials. That is why construction of the differentials allows us to write solution of the Riemann problem in closed form.

In this paper the Abelian differentials of the first kind have been constructed on Riemannian surface represented in the form of N copies of the one-point compactified complex plane without discs. Circumferences are identified in prescribed rule. As an example a double of multiply connected domain refers to such surfaces. Each multiply connected domain can be mapped conformally to circular domain [7]. Hence, if that conformal mapping is known, then the Abelian differentials of double of the origin domain are known too.

In order to construct the Abelian differentials we use the Poincare θ_2-series for a Kleinian group. Moreover, functional equations for analytic functions [9] and **R**-linear boundary value problem [10] are used. One can consider an Abelian differential as a solution of the simplest Riemann boundary value problem on a Riemannian surface. The last general problem on double of a multiply connected domain has been solved in [11].

Let us consider N numerated copies of the one-point compactified complex plane $\overline{C}$. Let $D_k := \left\{ z \in C, \left| z - a_k \right| < r_k \right\} \left(k = 1,2,\ldots,n \right)$ be mutually disjoint discs. Let for each number k it is possible to indicate two numbers $l(k)$ and $q(k)$ of the copies of $\overline{C}$ containing the disc with the number k. Everywhere in the paper we shall denote by l and q numbers of sheets corresponding to the number k. Moreover, we assume that l is odd and q is even numbers. For each l -th sheet exists the set I_l of the numbers of discs belonging to the l -th copy of $\overline{C}$. Let us consider the multiply connected circular domain

$$G_l := \left\{ points\ of\ the\ l-th\ sheet,\ not\ belonging\ to\ \bigcup_{m\in I_l}^{n} \overline{D_m} \right\}.$$

Let us identify the boundaries of all domains G_l and G_q along the circumferences $\partial D_k := \{t \in C, |t-a_k| = r_k\}$ oriented in the positive direction. For each number of circumference k only two domains G_l and G_q corresponding to k are identified. Let us suppose that after this procedure we get a closed Riemannian surface R. On each sheet with odd number we introduce the local coordinate z, on the sheet with even number - $\overline{z}$. Then the Riemannian surface R is oriented.

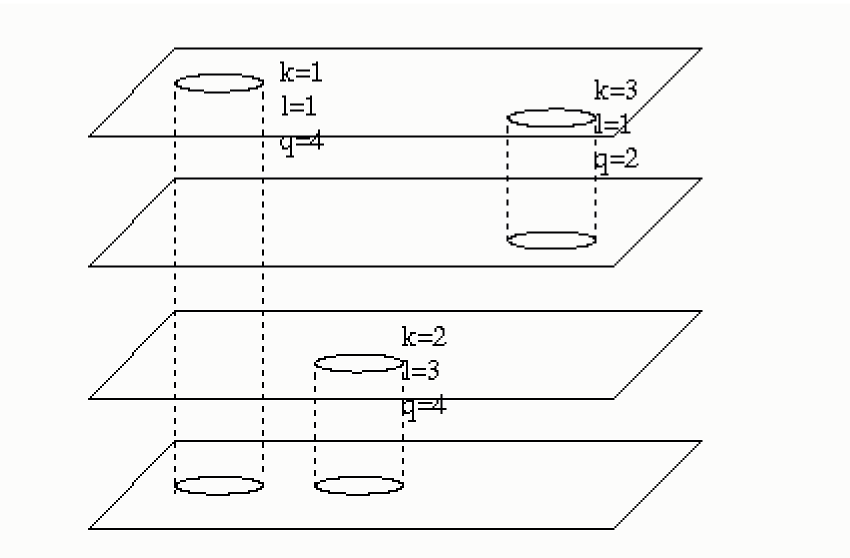

Fig.1. An example of the Rimannian surface in question

Let dw be an Abelian differential of the first kind on R. Let us denote

$$dw = dw_p(z),\ z\ belongs\ to\ the\ p-th\ sheet,\ p\ is\ odd,$$

$$dw = dw_p(\overline{z}),\ z\ belongs\ to\ the\ p-th\ sheet,\ p\ is\ even,$$

$p = 1, 2,..., N$. The section of dw on the p-th sheet can be written in the form

$$dw_p(z) = d\varphi_p(z) + \sum_{m\in I_p} A_{pm} \frac{dz}{z-a_m}, \ \ p \text{ is odd,}$$

$$dw_p(\bar{z}) = \overline{d\varphi_p(z)} + \overline{\sum_{m\in I_p} A_{pm} \frac{dz}{z-a_m}}, \ \ p \text{ is even.} \tag{1.1}$$

Here the function $\varphi_p(z)$ is analytic and single-valued in $\overline{G_p}$,

$$2\pi i A_{pm} = \int_{\partial D_m} dw_p, \ p = 1, 2, ..., N$$

is a period of dw_p along ∂D_m. The condition of analytic continuation of dw through ∂D_k (l and q correspond to k) has the form

$$dw_l(t) = dw_q(\bar{t}), \quad |t-a_k| = r_k, \ \ k = 1, 2, ..., n \tag{1.2}$$

Let us fix the branch of the function $\ln(z-a_k)$ in such a way that whole cut connecting the points $z = a_k$ and $z = \infty$ lies in the domain $D \cup D_k$, where $D := \overline{C} - \bigcup_{s=1}^{n} \overline{D_s} = \bigcap_{m=1}^{N} G_m$.

Integrating (1.2) and applying (1.1), we obtain

$$\varphi_l(t) = \overline{\varphi_q(t)} - f_{lk}(t) + \overline{f_{qk}(t)}, \quad |t-a_k| = r_k,, \tag{1.3}$$

where

$$f_{lk}(t) := \sum_{m\in I_k \neq k} A_{lm} \ln(t-a_m) + b_k, \qquad f_{qk}(t) := \sum_{m\in I_k \neq k} \overline{A_{qm}} \ln(t-a_m), \ k = 1, 2, ..., n.$$

Here $\sum_{m\in I_k \neq k} := \sum_{m\in I_k}$ with $m \neq k$, b_k is a constant . One can consider the equality (1.3) as the simplest boundary value problem on R [5]. Calculate the period

$$\int_{\partial D_k} dw(z) = \int_{\partial D_k} dw_l(z) = 2\pi i A_{lk}.$$

From other hand

$$\int_{\partial D_k} dw(z) = - \int_{\partial D_k} dw_q(\bar{z}) = 2\pi i \overline{A_{qk}}.$$

Then

$$A_{lk} = \overline{A_{qk}}\ , k = 1, 2,..., n. \qquad (1.4)$$

The sum of the residues of dw_p at infinity is equal to zero. Hence,

$$\sum_{m\in I_p} A_{pm} = 0 \quad , \ p = 1, 2,..., N. \qquad (1.5)$$

2.Reducing the problem (1.3) to a system of functional equations

According to scheme of [8, 11] let us rewrite the problem (1.3) in the form of a vector-matrix **R**-linear boundary value problem

$$\Phi(t) = \Phi_k(t) + Q_k\overline{\Phi_k(t)} - F_k(t), \quad |t-a_k| = r_k\ , \text{k = 1, 2,..., n,} \qquad (2.1)$$

where

$$\Phi(z) = \begin{pmatrix} \varphi_1(z) \\ . \\ . \\ . \\ \varphi_N(z) \end{pmatrix}, \quad \Phi_k(z) = \begin{pmatrix} \varphi_{1k}(z) \\ . \\ . \\ . \\ \varphi_{Nk}(z) \end{pmatrix}.$$

The uknown vector-functions $\Phi(z)$ and $\Phi_k(z)$ are analytic in D and D_k respectively and continuously differentiable in $\overline{D}, \overline{D_k}$. The known vector-function $F_k(t)$ has the *l*-th coordinate $f_{kl}(t)$, the *q*-th - of $f_{ql}(t)$, the rest coordinates are equal to zero. The matrix Q_k has the dimension *N x N* and consists of zeros except the coordinates (*l*,*q*) and (*q*,*l*), with 1. As above the numbers *l* and *q* correspond to the number of circumference *k*.

Let us show the equivalence of the problem (1.3) and (2.1). Let us fix the number of circumference *k*. If $\Phi(z), \Phi_k(z)$ satisfy (2.1), then

$$\varphi_s(t) = \varphi_{sk}(t),$$

for $s \neq l$ and *q*, i.e. $\varphi_s(t)$ is analytically continuied to G_s. For $s = l$

and $s = q$ we have the relations

$$\varphi_l(t) = \varphi_{lk}(t) + \overline{\varphi_{qk}(t)} - f_{lk}(t), \quad \varphi_q(t) = \varphi_{qk}(t) + \overline{\varphi_{lk}(t)} - f_{qk}(t), \quad |t-a_k| = r_k\,.$$

Therefore, the equality (1.3) holds. Conversly, let $\varphi_l(z)$ satisfy (1.3). Then the functions $\varphi_{lk}(z)$ are restored up to an additive complex constant from two Schwarz problems for the disc $|z-a_k|\le r_k$ [12]:

$$\operatorname{Re}[\varphi_{lk}(t)+\varphi_{qk}(t)]=\operatorname{Re}[\varphi_l(t)+f_{lk}(t)],\quad \operatorname{Im}[\varphi_{lk}(t)-\varphi_{qk}(t)]=\operatorname{Im}[\varphi_l(t)+f_{lk}(t)],\ |t-a_k|=r_k.$$

It is easily to check, that these functions satisfy (2.1).

Let us reduce the problem (2.1) to a system of functional equations. Let us introduce the vector-function

$$\Omega(z):=\Phi_k(z)-\sum_{m=1\neq k}^{n}Q_m\overline{\Phi_k(z_m^*)}-F_k(z),\ |z-a_k|\le r_k,\ k=1,2,\dots,n,$$

$$\Omega(z):=\Phi(z)-\sum_{m=1}^{n}Q_m\overline{\Phi_k(z_m^*)},\quad z\in\overline{D}.$$

Here $\sum\limits_{m=1\neq k}^{n}:=\sum\limits_{m=1}^{n}$ with $m\neq k$, $z_m^*:=r_m^2/\overline{(z-a_m)}+a_m$ is the inversion with respect to the circumference $|t-a_k|=r_k$. Let us show that $\Omega(z)$ is analytic in **C**. We have

$$\Omega^+(t)-\Omega^-(t)=\Phi_k(t)+F_k(t)-\Phi(t)-Q_k\overline{\Phi_k(t)}=0,\quad |t-a_k|=r_k.$$

Taking into account the principle of analytic continuation and the Liouville theorem we have

$$\Omega(z)=\Omega(w)=\Phi(w)-\sum_{m=1}^{n}Q_m\overline{\Phi_m(w_m^*)},$$

where w is a fixed point belonging to $\overline{D}-\{\infty\}$. From the definition of $\Omega(z)$ in D_k we obtain the following relations

$$\Phi_k(z)=\sum_{m=1\neq k}^{n}Q_m\left[\overline{\Phi_m(w_m^*)}-\overline{\Phi_m(w_m^*)}\right]-Q_k\overline{\Phi_k(w_k^*)}+F_k(z)+\Phi(w),\ |z-a_k|\le r_k,\ k=1,2,\dots,n. \tag{2.2}$$

These relations constitute a system of n x N linear scalar functional equations for n unknown vector-functions $\Phi_k(z)$ (k = 1, 2,..., n), which are analytic in D_k and are continuously differentiable in $\overline{D_k}$. Introduce new notations for the most important components of $\Phi_k(z)$. Let

$$\psi_k(z) = \varphi_{lk}(z), \quad \omega_k(z) = \varphi_{qk}(z), \quad |z - a_k| \le r_k .$$

Remark. *Further we shall write one relation on* $\psi_k(z)$ *and* $\omega_k(z)$ *instead of two ones.*

The auxiliary functions $\varphi_{sk} \quad (s \ne l \text{ and } q)$ are represented by $\psi_k(z)$ and $\omega_k(z)$ in (2.2). The *l*-th component of (2.2) for each *k* takes the form

$$\psi_k(z) = \sum_{m \in I_l \ne k} \left[\overline{\omega_m(w_m^*)} - \overline{\omega_m(w_m^*)} \right] - \overline{\omega_k(w_k^*)} + f_{lk}(z) + \varphi_l(w), \; |z - a_k| \le r_k, k = 1, 2, ..., n. \quad (2.3)$$

We keep in our mind in (2.3) the analogous relations, when $\psi_k(z)$ and $\omega_k(z)$, l and q are changed by places. The relations (2.3) can be considered as a system of $2n$ scalar functional equations with respect to $2n$ unknown functions $\psi_k(z)$ and $\omega_k(z)$. The system (2.2) and (2.3) are equivalent. From the definition of $\Omega(z)$ we have the relations

$$\varphi_l(z) = \sum_{m \in I_l} \left[\overline{\omega_m(w_m^*)} - \overline{\omega_m(w_m^*)} \right] + \varphi_l(w), \quad z \in \overline{G_l}.$$

According to (1.1) we need only the derivative

$$\varphi_l'(z) = \sum_{m \in I_l} \left[\overline{\omega_m(w_m^*)} \right]', \quad z \in \overline{G_l} \quad (2.4)$$

to calculate the differentials.

3.Solution of the system of functional equations.

Let us prove convergence of the successive approximation method for the system (2.3). We shall use some auxiliary assertions.

Consider the Banach space C consisting of functions continuous on $\bigcup_{k=1}^{n} \partial D_k$. The norm $\|\Psi\| := \max_{0\le k\le n} \max_{\partial D_k} \left[\sum_{s=1}^{N} |\Psi_s(t)|^2 \right]^{1/2}$, where $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_N)^T$. Introduce the subspace $C^+ \subset C$, which consists of vector-functions analytic in all D_k . Differentiate the system (2.3)

$$\Psi_k(z) = -\sum_{m\in I_l \ne k} Q_m \left[\overline{z_k^*}\right]' \overline{\Psi_m(z_m^*)} + F_k'(z), \ |z-a_k| \le r_k . \tag{3.1}$$

Rewrite the last system in the form of equation

$$\Psi(z) = A\Psi(z) + F'(z) \tag{3.2}$$

in the space C^+ , where the operator A is defined by the right-hand part of the system (3.1), $\Psi(z) := \Psi_k(z), \quad F(z) := F_k(z)$, when $|z-a_k| \le r_k$; $\Psi, F' \in C^+$.

Lemma 1. [8]. *The equation (3.2) is a Fredholm equation in* C^+ .

Lemma 2. *The homogeneous equation (3.2)* $(F'(z) \equiv 0)$ *has zero solution only.*

P r o o f. Let us consider the differentiated homogeneous system (2.3) corresponding to homogeneous equation (3.2)

$$\psi_k'(z) = \sum_{m\in I_l \ne k} \left[\overline{z_k^*}\right] \left[\overline{\omega_m(w_m^*)}\right]', \ |z-a_k| \le r_k . \tag{3.3}$$

As ealier we write only one equation. If the system (3.3) has only trivial solution, then homogeneous equation (3.2) has only trivial solution too.

Integrating the system (3.3) we obtain

$$\psi_k(z) = \sum_{m\in I_l \ne k} \left[\overline{\omega_m(w_m^*)}\right] + \gamma_{lk}, \ \ |z-a_k| \le r_k ,$$

where γ_{lk} is a constant of integration. If ω_k and ψ_k is a solution of the last system , then the functions

$$\varphi_l(z) = -\sum_{m\in I_l} \overline{\omega_m(w_m^*)}, \quad z \in \overline{G_l},$$

satisfy the conditions

$$\psi_k(t) = \varphi_l(t) + \overline{\omega_k(t)} + \gamma_{lk}, \quad \omega_k(t) = \varphi_q(t) + \overline{\psi_k(t)} + \gamma_{qk}, \quad |t - a_k| = r_k .$$

Hence, the functions $\varphi_l(t)$ and $\varphi_q(t)$ are related by equalities

$$\varphi_l(t) = \overline{\varphi_q(t)} + \overline{\gamma_{qk}} - \gamma_{lk}, \quad |t - a_k| = r_k .$$

Differetiating this relation we have

$$d\varphi_l(t) = d\overline{\varphi_q(t)}, \quad |t - a_k| = r_k .$$

The last relations define an Abelian differential of the fist kind on the surface R. But the functions $\varphi_l(t)$ are single-valued in $\overline{G_l}$. Therefore, if we assume that ∂D_k are canonical sections of R, then the period

$$\int_{\partial D_k} d\varphi_l(z)dz = 0, \quad k \in I_l .$$

And we immeadetly obtain, that $\varphi_l(z)$ is a constant. Then $\omega_m(z)$ is a constant too, and $\omega_m'(z) = 0$.

The lemma is proved.

Let us consider the **R**-linear boundary value problem

$$\Phi(t) = \Phi_k(t) - \lambda Q_k \overline{\Phi_k(t)} - \gamma_k, \quad |t - a_k| = r_k, \qquad k = 1, 2,..., n, \quad (3.4)$$

where the unknown vector-functions $\Phi(z), \Phi_k(z)$ are analytic in D, D_k respectively and are continuously differentiable in $\overline{D}, \overline{D_k}$. Here λ is a constant, γ_k is a constant vector.

Lemma 3. *The problem (3.4) for* $|\lambda| < 1$ *has constant solutions only.*

P r o o f. We shall use the idea of Bojarski [10]. Let us put

$$U(z) := \Phi(z), \quad z \in D, \quad U(z) := \Phi_k(z) - \lambda Q_k \overline{\Phi_k(z)} - \gamma_k, \quad z \in D_k .$$

Then the vector-function $U(z)$ is a solution of the following partial differential equation

$$U_{\bar z} + Q\overline{U_z} = 0, \quad z \in \overline{C} - \bigcup_{k=1}^{n} \partial D_k, \qquad (3.5)$$

where $Q=0, \quad z\in D,\ Q=\lambda Q_k, \quad z\in D_k$. Let us check that the system (3.5) is elliptic. The determinator of that system is

$$\Delta=\det\begin{pmatrix} I-Q & I+Q \\ I-Q & -I-Q \end{pmatrix}\neq 0 \quad \Leftrightarrow \quad \Delta_1 := \det M \neq 0, \text{where} \quad M := \begin{pmatrix} I & -Q \\ -Q & I \end{pmatrix}.$$

Here the second matrix M is obtained from the first by summation and reduction of columns. We have $\Delta_1 \neq 0$, because

$$M^{-1}=\begin{pmatrix} \left(I-Q^2\right)^{-1} & Q\left(I-Q^2\right)^{-1} \\ Q\left(I-Q^2\right)^{-1} & \left(I-Q^2\right)^{-1} \end{pmatrix}.$$

exists. It can be verified by direct calculation of MM^{-1}. Let us show the existence of $\left(I-Q^2\right)^{-1}$. The matrix $\left(I-Q^2\right)$ is equal to I except the l-th and the q-th elements for $z\in D_k$, which equal to $1-\lambda^2$. The inequality $|\lambda|<1$ implies that the matrix $\left(I-Q^2\right)$ is diagonal and inversable. So we have proved, that (3.5) is elliptic. The condition $U^+=U^-$ is valid on ∂D_k, moreover, $U^{\pm}\in L_2\left(\partial D_k\right)$. Hence, (3.5) holds in $\overline{C}$. Taking into account the general Liouville theorem we have the equality $U = const$. Therefore, the problem (3.4) for $|\lambda|<1$ has constant solutions only.

The lemma is proved.

Lemma 4. *The equation (3.2) has one and only one solution in C^+. This solution can be found by the method of successive approximations in C^+.*

P r o o f. Let us rewrite the system (3.1) on ∂D_k in the form of integral equations

$$\Psi_k(t)=-\sum_{m=1\neq k}^{n}\left[\overline{t_m^*}\right]' Q_m \overline{\frac{1}{2\pi i}\int_{\partial D_m}\frac{\Psi_m\left(\tau_m^*\right)}{\tau-t_m^*}}+F_k'(t), \quad |t-a_k|=r_k .$$

It can be written as the equation in C

$$\Psi(t)=A\Psi(t)+F'(t). \tag{3.6}$$

The integral operator is compact in C. The operator of multiplication on the matrix $\left[\overline{t_m^*}\right]' Q_m$ and the operator of complex conjugation are bounded in C. Hence, A is a compact operator in C. The Cauchy integral property implies if Ψ is a solution of (3.2) in C, then $\Psi \in C^+$. Therefore, the equation (3.2) in C and the equation (3.6) in C^+ are equivalent when $F' \in C^+$. It follows from Lemma 2 that the homogeneous equation $\Psi = A\Psi$ has only zero solution. Then the Fredholm theorem implies that the system (3.6) or the system (3.1) has the unique solution.

Let us demonstrate convergence of the method of successive approximations. It is sufficient to prove the inequality $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A. The compact operator A has a spectrum consisting of eigenvalues [13]. The inequality $\rho(A) < 1$ is observed iff there exists a complex number λ such that $|\lambda| \le 1$ and the equation

$$\Psi(t) = \lambda A \Psi(t)$$

has zero solution only. The equation can be written in the form

$$\Psi_k(z) = -\lambda \sum_{m \ne k} Q_m \left[\overline{z_k^*}\right]' \overline{\Psi_m\left(z_m^*\right)}, \quad |z - a_k| \le r_k. \tag{3.7}$$

Let $|\lambda| < 1$. Then, we introduce the vector-function

$$\Omega(z) = -\lambda \sum_{m=0}^{n} Q_m \left[\overline{z_k^*}\right]' \overline{\Psi_m\left(z_m^*\right)},$$

which is analytic in $\overline{D}$. From (3.7) we have

$$\Omega(t) = \Psi_k(t) - \lambda \sum_{m=0}^{n} \left[\overline{t_k^*}\right]' Q_k \overline{\Psi_k\left(t_k^*\right)}, \quad |t - a_k| = r_k.$$

By integrating the relations we obtain the following **R**-linear boundary value problem

$$\Phi(t) = \Phi_k(t) - \lambda Q_k \overline{\Phi_k(t)} + \gamma_k, \quad |t - a_k| = r_k.$$

Here $\Phi'(z) = \Omega(z)$, $\Phi_k{}'(z) = \Psi_k(z)$, γ_k are arbitrary constant vectors. It follows from Lemma 3 that the **R**-linear boundary value problem has constant solutions only. Then $\Phi'(z) = \Phi_k{}'(z) = 0$.

Let $|\lambda| = 1$. Then, by changing the variable $z = \sqrt{\lambda} Z$ we reduce the system (3.7) to the same system with λ = 1, where $a_k = \sqrt{\lambda} A_k$ and $\Omega_k(Z) := \Psi_k(z)$. It follows from Lemma 3 that $\Omega_k(Z) = \Psi_k(z) = 0$. Hence, ρ($A$) < 1.

This inequality proves the lemma.

Introduce the mappings

$$z^*_{k_m k_{m-1} \ldots k_1} := \left(z^*_{k_{m-1} \ldots k_1}\right)^*_{k_m}.$$

There are no equal neighbor numbers in the sequence $k_1, k_2, \ldots, k_m$. When m is even we have Moebius transformations on z. If m is odd we have transformations on $\overline{z}$.

Theorem 1. *The system of functional equations*

$$\psi_k{}'(z) = \sum_{m \in I_l \neq k} \left[\overline{z_k^*}\right]' \left[\overline{\omega_m\left(w_m^*\right)}\right] + f_{lk}{}'(z), \quad |z - a_k| \leq r_k, \quad k = 1,2,\ldots,n, \tag{3.8}$$

has the unique solution

$$\psi_k{}'(z) = f_{lk}{}'(z) + \sum_{k_1 \in I_l \neq k} \left[\overline{f_{q_1 k_1}\left(z^*_{k_1}\right)}\right]' + \sum_{k_1 \in I_l \neq k} \sum_{k_2 \in I_q \neq k_1} \left[f_{l_{21} k_2}\left(z^*_{k_2 k_1}\right)\right]' +$$

$$\sum_{k_1 \in I_l \neq k} \sum_{k_2 \in I_q \neq k_1} \sum_{k_3 \in I_l \neq k_2} \left[\overline{f_{q_3 k_3}\left(z^*_{k_3 k_2 k_1}\right)}\right]' + \ldots, \quad |z - a_k| \leq r_k.$$

The proof follows from Lemma 4. Because the equality (3.8) is the l-th coordinate of the vector equality (3.1) for fixed k.

The theorem is proved.

Remark. *The convergence in* C^+ *means the uniform convergence.*

From (2.4) we have

$$\varphi_k^{'}(z) = \sum_{k_0 \in I_l} \left[\overline{\omega_{k_0}\left(z_{k_0}^*\right)} \right]^{'} = \sum_{k_0 \in I_l} \left[\overline{f_{q_0 k_0}\left(z_{k_0}^*\right)} \right]^{'} + \sum_{k_0 \in I_l} \sum_{k_1 \in I_l \neq k_0} \left[f_{l_{21} k_2}\left(z_{k_1 k_0}^*\right) \right]^{'} + \tag{3.9}$$

$$\sum_{k_0 \in I_l} \sum_{k_1 \in I_l \neq k} \sum_{k_2 \in I_q \neq k_1} \left[f_{q_3 k_3}\left(z_{k_3 k_2 k_1}^*\right) \right]^{'} + \ldots, \quad z \in \overline{G_l}.$$

The formula

$$\left[f_{lk}\left(\alpha(z)\right) \right]^{'} = \sum_{m \in I_l \neq k} A_{lk} \frac{\alpha^{'}(z)}{\alpha(z) - a_m}$$

follows from the definitions of f_{lk}. So we have the final

Theorem 2. *All Abelians differentials of the first kind on the Riemannian surface* R *have the form (1.1), (3.9), where the constants* A_{lm} *are related by (1.4), (1.5).*

Let us study the number of the constants A_{lm} corresponding to linear independent differentials (1.1). According to the general theory [1,2] that number is equal to the genius ρ(R). Let us consider the surface Q homeomorphic to R, when instead of the discs we have cuts gluied in the same rule. Let V be a branch index of Q [2]. It is easily seen, that $V = 2n$, where $2n$ is the number of ends of the cuts, 1 is the order of branching. The following formula

$$V = 2(N + \rho(Q) - 1)$$

holds [2]. Then ρ(R) = ρ(Q) = $n - N + 1$. The number of constants A_{lm} is equal to $2n$. After (1.4) that number reduces on n. Hence, from N relations (1.5) we can choose (N - 1) linear independent relations.

References

[1] M.Schiffer, D.K.Spencer, *Functionals on finite Rimaniannian surfaces*, Moscow, IL 1957

[2] G.Springer, *Introduction to Riemannian Surfaces,* Addison-Wesley, Reading, MA 1957

[3] N.I.Zhukova, E.I.Zverovich, *Riemannian surfaces of regular 2n-polygons*, Preprint VINITI N 887-76, 1976, Minsk-Moscow (in Russian).

[4] Setko E. A., E.I.Zverovich, *Riemannian surfaces of regular 4n -polygons*, Preprint VINITI N 1845-B90, 1990, Minsk-Moscow (in Russian).

[5] E.I.Zverovich, *Boundary value problems of the theory of analytic functions in Hoelder clases on Riemannian surfaces*, Uspehi Mat. Nauk 26(1971), 113-179 (in Russian).

[6] H.Poincare, *Ouevres*, Gauthier-Villar, Paris t.2, 1916; t.4, 1950

[7] Golusin G.M., *Geometrische Funktionentheorie*, Verlag, Berlin 1957

[8] V.V.Mityushev, *On solution of boundary R-value problem for complex cotour*, Preprint VINITI N 158-B86, 1986, Minsk-Moscow (in Russian).

[9] Kuczma M., *Functional equations in a single variable*, PWN, Warszawa,1968.

[10] B.Bojarski, *On generated Hilbert boundary value problem*, Soobszczenia AN Gr SSR 25 (1960), 385-390 (in Russian).

[11] V.V.Mityushev, *Riemann problem on double of a multiply connected circular region,* Ann.Pol.Math., 1997 (to appear)

[12] F.D.Gakhov, *Boundary value problems*, Nauka, Moscow 1977 (in Russian)

[13] Krasnoselski MA, Vainikko, GM, Zabreiko, PP, Rutitski, Ya. B., Stetsenko,. V.Ya. *Approximate Solution of Operator Equations*. Nauka, Moscow 1969 (in Russian).