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# Effective elastic properties of random two-dimensional composites

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Consider 2D two-phase random composites with circular inclusions of concentration $f$. New analytical formulae for the effective constants are deduced up to $O(f^4)$ for macroscopically isotropic composites. It is shown that the second order terms $O(f^2)$ do not depend on the location of inclusions whilst the third order terms do. This implies that the previous analytical methods (effective medium approximation, differential scheme, Mori–Tanaka approach and so forth) can be valid at most up to $O(f^3)$ for macroscopically isotropic composites. First, the local elastic field for a finite number $n$ of inclusions arbitrarily located on the plane are found by a method of functional equations. Further, the limit $n \to \infty$ yields conditionally convergent series defined by the Eisenstein summation method. One of the series for periodic composites is the famous lattice sum $S_2 = \pi$ deduced by Rayleigh for a conductivity problem.



## 1. Introduction

Analytical formulae for the effective properties of random composites are of great interest in the fundamental science and engineering applications. In the present paper, we deduce formulae for 2D two-phase elastic composites with equal unidirectional fibers of circular section with isotropic components. For definiteness, the discussed composites are assumed to be macroscopically isotropic. Formulae for dilute composites in the first order approximation in the concentration $f$ were deduced by various self consistent methods shortly addressed below as SCM (effective medium approximation, differential scheme, Mori–Tanaka approach and so forth (Kanaun and Levin, 2008)). All these formulae are equivalent up to $O(f^2)$ and the effective constants $\mu_e$ and $k_e$ can be written in the form (Bertoldi et al., 2007)

$$\frac{\mu_e}{\mu} = 1 + (\kappa+1)\frac{\frac{\mu_1}{\mu} - 1}{\kappa\frac{\mu_1}{\mu} + 1} f + O(f^2), \tag{1}$$

$$\frac{k_e}{k} = 1 + (\kappa+1)\frac{\frac{\mu_1}{\mu} - \frac{\kappa+1}{\kappa_1-1}}{\kappa_1 - 1 + 2\frac{\mu_1}{\mu}} f + O(f^2), \tag{2}$$

where $\mu$ denotes the shear modulus of matrix; $\kappa = 3 - 4\nu$ for the plane strain ($\kappa = \frac{3-\nu}{1+\nu}$ for the plane stress); $\nu$ is the Poisson's ratio; $\mu_1$ and $\kappa_1$ denote the corresponding elastic constants for inclusions. Analogous formulae were deduced for other shapes and for multiphase composites (see Kanaun and Levin (2008) and works cited therein). A lot of efforts have been applied to extend formulae (1) and (2) to high orders by use of the SCM without a justification.

In the present paper, we develop a rigorous method to extend (1) and (2) to higher order concentrations. Though the method practically has no restrictions, here, we explicitly present the results only up to $O(f^4)$. We take the minimal order $O(f^3)$ of symbolic computations in order to explicitly demonstrate the principal impossibility to deduce high order formulae without using of microstructure information given in terms of the high order statistical description (see $n$-point correlation functions (Torquato, 2002) and the basic $e$-sums (Czapla et al., 2012)). Advanced symbolic computations for higher orders of $f$ and comparison with pure numerical results will be presented in a separate publication.

Such a study had already been performed for the conductivity problem. It was shown in Mityushev and Adler (2002) and Mityushev and Rylko (2013) that all SCM yield actually only one formula, the famous Clausius–Mossotti approximation, which can be applied only within the first order approximation. Other formulae can be obtained from it by elementary manipulations within the accuracy $O(f)$. It was also shown that for macroscopically isotropic composites the second order term for the effective conductivity does not depend on the location of inclusions whilst the

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third order term does. This implies that any SCM is valid up to $O(f^2)$ in general case and up to $O(f^3)$ for macroscopically isotropic composites. In the present paper, we demonstrate that the same result holds for 2D elasticity problem.

As in Mityushev and Rylko (2013), the main difficulty is the proper treatment of the conditionally convergent series (integrals) arisen in the seminal papers by Rayleigh (1892) and discussed in Batchelor and Green (1972) and Jeffrey (1973) (see the recent review Brady et al. (2006)). Rayleigh (1892) discussed regular composites (one inclusion per periodicity cell) and justified the famous formula $S_2 = \pi$ for the conditionally convergent series (55). Batchelor and Green (1972) and others (the most advanced is the paper Wang et al. (2003)) used $n$-particle interactions ($n$ was taken as a finite number, not in symbolic form). Roughly speaking, Batchelor's approach was based on the finite approximation of the conditionally convergent series similar to Rayleigh's series (55). Rayleigh's method (Rayleigh, 1892) to the elastic problem for regular arrays was extended in the papers Drummond and Tahir (1984), Greengard and Helsing (1998) and Movchan et al. (1997). One of the main step of these works was the proper treatment of the conditionally convergent series (59). For random composites, the series (45) and (47) were investigated in Mityushev (1999) and Czapla et al. (2012), but the series (46), (48), (49) and (59) for the hexagonal array were not studied. It is worth noting that the conditionally convergent series (55) and (59) for regular arrays, (45)–(49) for random composites are the key values for high order formulae (see Czapla et al. (2012) and Gluzman and Mityushev (2015) for the effective conductivity).

The homogenization theory of random media deals with *uniqueness and existence* of the RVE (representative volume element) and of the effective constants. The direct $\varepsilon$-method yields analytical formulae only for general 1D and some 2D composites (Jikov et al., 1994). Variational methods of the homogenization theory yield the theory of bounds (Milton, 2002). Some of them can be written in analytical form as the Hashin–Strikman bounds (3)–(5). Bounds are useful when microstructure of composites is not known. If it is known in the form of statistical information, the effective properties of composites can be precisely determined as the mathematical expectation of the effective constants of the statistically representative cells. It was done in Czapla et al. (2012), Mityushev and Nawalaniec (2015) and Gluzman and Mityushev (2015) for 2D conductivity problems. In the present paper, we extend the results (Czapla et al., 2012) to 2D elastic composites with arbitrary locations of circular identical inclusions. Any plane shape can be approximated by an appropriate packed disks. This implies that the obtained results can be extended to other shapes, and it will be discussed in a separate paper with advanced symbolic computations.

Construction of the RVE is a separate question not discussed here. We refer to Mityushev (2006) and Rylko (2014) where a computationally effective method was proposed and to Czapla et al. (2012), Mityushev and Nawalaniec (2015) and Kurtyka et al. (2015) where the method was developed and applied to casting stir processes.

This paper is organized as follows. In Section 2, we discuss the Hashin–Shtrikman bounds Hashin and Shtrikman (1962) and the MMM principle by Hashin (1983). Further, we demonstrate that Hashin's physical approach is in agree with the homogenization theory of random media developed in Golden and Papanicolaou (1983), Jikov et al. (1994) and Telega (2005). Section 3 is devoted to application of the method of functional equations in order to obtain the local elastic fields in analytical form for media with a finite number of inclusions $n$. The obtained formulae are used in Section 4 to calculate the average elastic constants. Section 5 contains subtle mathematical discussion of the limit formulae as $n \to \infty$. One can consider our investigations as a rigorous mathematical treatment of the problem of infinite integral discussed in Rayleigh (1892) and Jeffrey (1973) and their extension to random composites. Analytical formulae for the effective elastic constants are written in Section 6. We demonstrate that our approach is in accordance with the homogenization theory and serves as a mathematical realization of the MMM principle by Hashin. Concluding remarks are collected in Section 7. Some long formulae and expressions are written in Supplementary which contains also a high order formula for the hexagonal regular array of disks.

## 2. The Hashin–Shtrikman bounds and Hashin's approach

Any expression for the effective constants must obey the Hashin–Strikman bounds. Let $\mu_1 \geq \mu$ and $k_1 \geq k$. Then, we have (Hashin, 1983; Hashin and Shtrikman, 1962)

$$\mu^- \leq \mu_e \leq \mu^+, \quad k^- \leq k_e \leq k^+, \tag{3}$$

where

$$k^- = k + \frac{f}{\frac{1}{k_1-k} + \frac{1-f}{k+\mu}}, \quad k^+ = k_1 + \frac{1-f}{\frac{1}{k-k_1} + \frac{f}{k_1+\mu_1}}, \tag{4}$$

$$\mu^- = \mu + \frac{f}{\frac{1}{\mu_1-\mu} + \frac{(1-f)(k+2\mu)}{2\mu(k+\mu)}}, \quad \mu^+ = \mu_1 + \frac{1-f}{\frac{1}{\mu-\mu_1} + \frac{f(k_1+2\mu_1)}{2\mu_1(k_1+\mu_1)}}. \tag{5}$$

Formulas (1) and (2) coincides up to $O(f^2)$ with the low HS bounds $\mu^-$ and $k^-$ (Bertoldi et al., 2007). The Padé approximation coincides with the low HS (see details in Chapters 23, 27 of the book Milton (2002)).

The effective constants can be estimated by two methods leading to the same result. The first method is based on the doubly periodic problems. It was proposed and constructively applied by using of integral equations due to Filshtinsky (Grigolyuk and Fil'shtinskij, 1970; 1992; Helsing, 1995). Pure numerical methods (FEM etc) leading to numerical results are applied by many authors. We omit a long discussion concerning numerical methods and deal only with analytical methods leading to analytical formulae.

In the present paper, we follow the second method consisting of two steps. A problem with a finite number of inclusions $n$ is solved at the first step. Further, the limit of the obtained solution is investigated as $n \to \infty$. This formal mathematical approach is consistent with the MMM principle by Hashin based on the investigation of the homogenization problem in the scales (Hashin, 1983)

$$\text{MICRO} \ll \text{MINI} \ll \text{MACRO} \tag{6}$$

Following Hashin (1983) microstructure of composites is analyzed on the level MICRO. Further, a representative volume element (RVE) is introduced during the passage from MICRO to MINI. The macroscopic constants on the level MACRO are constructed by means of the RVE. We refer to the review Hashin (1983) for the extended physical discussion of the MMM principle.

Many discussions on the physical level can be found in literature which can be considered as using of the MMM principle for multilevel structures by repeated application of the scheme (6) and introduction mesoscales. Some conceptions skip the level MINI and treat homogenization as "periodization", for instance, approximate the structure of composite by a periodic material with one inclusion per cell. Such a method leads to correct results for regular composites.

The mathematical homogenization theory of random media (Golden and Papanicolaou, 1983; Jikov et al., 1994; Telega, 2005) corresponds to Hashin's lines and shortly outlined below. First, it is assumed that a random field which describes microstructure on

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the level MICRO is statistically homogeneous. This is a mathematical assumption (axiom) accepted in the theory of random composites. This assumption yields existence of the RVE which represents the composite on the MINI level. This RVE is the benchmark of homogenization. Further, it is proved that the composite can be homogenized, i.e., the corresponding PDE converges to the homogenized equation with constant coefficients called the effective constants. It is worth noting that the composite is not necessary periodic. But one can consider a periodic material on the level MINI when RVE-cells form a periodic structure. Though all the RVEs can be different internally, each RVE obeys the same statistical distribution. Hence, any RVE yields the same effective constants on the level MACRO. Therefore, for any statistically homogeneous field there exist a set of the statistically same RVEs which forms the class of equivalence. The investigation of the stochastic composite can be simplified if we fix a RVE from the class of equivalence and consider a periodic composite represented by the RVE.

Thus, the homogenization theory justifies the physical MMM principle by Hashin (6). As it is noted in Introduction, the homogenization theory of random media deals with the rigorous mathematical definition of the effective constants and refers to the mathematical quantitative methods of existence and uniqueness. How to compute these effective constants is a separate question. In the next sections, we describe such an analytical constructive method for 2D elastic composites with circular inclusions.

## 3. Method of functional equations for local fields

We begin our study with a finite number $n$ of inclusions on the infinite plane. This number $n$ is given in symbolic form that allows us further to pass to the limit $n \to \infty$. Mutually disjoint disks $D_k := \{z \in \mathbb{C} : |z - a_k| < r\}$ $(k = 1, 2, ..., n > 1)$ are considered in the complex plane $\mathbb{C}$ of the variable $z = x + iy$. Let $D := \mathbb{C} \cup \{\infty\} \setminus \left(\cup_{k=1}^{n} D_k \cup \partial D_k\right)$, where $\partial D_k := \{t \in \mathbb{C} : |t - a_k| = r\}$. We assume that $\partial D_k$ are orientated in clockwise sense. Hereafter, we use the letter $t$ for a complex variable on a curve and $z$ in a domain.

The component of the stress tensor can be determined by the Kolosov–Muskhelishvili formulae (Muskhelishvili, 1966)

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\mathrm{Re}\ \varphi_k'(z), & z \in D_k, \\ 4\mathrm{Re}\ \varphi_0'(z), & z \in D, \end{cases} \tag{7}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = \begin{cases} -2\left[z\overline{\varphi_k''(z)} + \overline{\psi_k'(z)}\right], & z \in D_k, \\ -2\left[z\overline{\varphi_0''(z)} + \overline{\psi_0'(z)}\right], & z \in D, \end{cases}$$

where Re denotes the real part and the bar the complex conjugation. Let $\begin{pmatrix} \sigma_{xx}^\infty & \sigma_{xy}^\infty \\ \sigma_{yx}^\infty & \sigma_{yy}^\infty \end{pmatrix}$ be the stress tensor applied at infinity. Following (Muskhelishvili, 1966) introduce the constants

$$B_0 = \frac{\sigma_{xx}^\infty + \sigma_{yy}^\infty}{4}, \quad \Gamma_0 = \frac{\sigma_{yy}^\infty - \sigma_{xx}^\infty + 2i\sigma_{xy}^\infty}{2}. \tag{8}$$

Then,

$$\varphi_0(z) = B_0 z + \varphi(z), \quad \psi_0(z) = \Gamma_0 z + \psi(z), \tag{9}$$

where $\varphi(z)$ and $\psi(z)$ are analytical in $D$ and bounded at infinity. The functions $\varphi_k(z)$ and $\psi_k(z)$ are analytical in $D_k$ and twice differentiable in the closures of the considered domains.

The perfect bonding at the matrix-inclusion interface can be expressed by two equations (Muskhelishvili, 1966)

$$\varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} = \varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}, \tag{10}$$

$$\kappa_1\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)} = \frac{\mu_1}{\mu}\left(\kappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}\right). \tag{11}$$

The problem (10) and (11) is the classic boundary value problem of the plane elasticity. It was discussed in many works by various methods (Helsing, 1995; Linkov, 2002; Mogilevskaya et al., 2013; 2012; Muskhelishvili, 1966). Below, we concentrate our attention to its analytical solution.

Introduce the new unknown functions

$$\Phi_k(z) = \left(\frac{r^2}{z - a_k} + \overline{a_k}\right)\varphi_k'(z) + \psi_k(z), \ |z - a_k| \le r,$$

analytic in $D_k$ except the point $a_k$, where its principal part has the form $r^2(z - a_k)^{-1}\varphi_k'(a_k)$.

Let $z_{(k)}^* = r^2\left(\overline{z - a_k}\right)^{-1} + a_k$ denote the inversion with respect to the circle $\partial D_k$. If a function $f(z)$ is analytic in $|z - a_k| < r$, then $\overline{f(z_{(k)}^*)}$ is analytic in $|z - a_k| > r$. The problem (10) and (11) was reduced in Mityushev and Rogosin (1999) (see Eqs. (5.6.11) and (5.6.16) in Chapter 5), (Mityushev, 2000) to the system of functional equations

$$\begin{aligned}\left(\frac{\mu_1}{\mu} + \kappa_1\right)\varphi_k(z) = {}& \left(\frac{\mu_1}{\mu} - 1\right)\sum_{m \ne k}\left[\overline{\Phi_m(z_{(m)}^*)} - (z - a_m)\overline{\varphi_m'(a_m)}\right] \\ & - \left(\frac{\mu_1}{\mu} - 1\right)\overline{\varphi_k'(a_k)}(z - a_k) + \frac{\mu_1}{\mu}(1 + \kappa)B_0 z \\ & + p_0, \ |z - a_k| \le r, \ k = 1, 2, ..., n,\end{aligned} \tag{12}$$

$$\begin{aligned}\left(\kappa\frac{\mu_1}{\mu} + 1\right)\Phi_k(z) = {}& \left(\kappa\frac{\mu_1}{\mu} - \kappa_1\right)\sum_{m \ne k}\overline{\varphi_m(z_{(m)}^*)} + \left(\frac{\mu_1}{\mu} - 1\right) \\ & \times \sum_{m \ne k}\left(\frac{r^2}{z - a_k} + \overline{a_k} - \frac{r^2}{z - a_m} + \overline{a_m}\right)\left[\left(\overline{\Phi_m(z_{(m)}^*)}\right)' - \overline{\varphi_m'(a_m)}\right] \\ & + \frac{\mu_1}{\mu}(1 + \kappa)B_0\left(\frac{r^2}{z - a_k} + \overline{a_k}\right) + \frac{\mu_1}{\mu}(1 + \kappa)\Gamma_0 z + \omega(z), \\ & |z - a_k| \le r, \ k = 1, 2, ..., n.\end{aligned} \tag{13}$$

where

$$\omega(z) = \sum_{k=1}^{n}\frac{r^2 q_k}{z - a_k} + q_0, \tag{14}$$

$q_0$ is a constant and

$$q_k = \varphi_k'(a_k)\left((\kappa - 1)\frac{\mu_1}{\mu} - (\kappa_1 - 1)\right) - \overline{\varphi_k'(a_k)}\left(\frac{\mu_1}{\mu} - 1\right), \quad k = 1, 2, ..., n. \tag{15}$$

The unknown functions $\varphi_k(z)$ and $\Phi_k(z)$ $(k = 1, 2, \ldots, n)$ are related by $2n$ Eqs. (12) and (13). One can see that the functional equations do not contain integral operators but contain compositions of $\varphi_k(z)$ and $\Phi_k(z)$ with inversions. These compositions define compact operators in a Banach space (Mityushev and Rogosin, 1999).

The functions $\varphi(z)$ and $\psi(z)$ are expressed through $\varphi_k(z)$ and $\psi_k(z)$ by formulae

$$\begin{aligned}\frac{\mu_1}{\mu}(1 + \kappa)\varphi(z) = {}& \left(\frac{\mu_1}{\mu} - 1\right)\sum_{m=1}^{n}\left[\overline{\Phi_m(z_{(m)}^*)} - (z - a_m)\overline{\varphi_k'(a_k)}\right] \\ & + p_0, \ z \in D,\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\mu_1}{\mu}(1 + \kappa)\psi(z) = {}& \omega(z) - \left(\frac{\mu_1}{\mu} - 1\right)\sum_{m=1}^{n}\left(\frac{r^2}{z - a_m} + \overline{a_m}\right) \\ & \times\left[\left(\overline{\Phi_m(z_{(m)}^*)}\right)' - \overline{\varphi_m'(a_m)}\right] + \left(\kappa\frac{\mu_1}{\mu} - \kappa_1\right) \\ & \times\sum_{m=1}^{n}\overline{\varphi_m(z_{(m)}^*)}, \ z \in D.\end{aligned} \tag{17}$$

We are looking for the complex potentials $\varphi_k$ and $\psi_k$ up to $O(r^6)$ in the form

$$\varphi_k(z) = \varphi_k^{(0)}(z) + r^2\varphi_k^{(1)}(z) + r^4\varphi_k^{(2)}(z) + O(r^6) \tag{18}$$

and

$$\psi_k(z) = \psi_k^{(0)}(z) + r^2\psi_k^{(1)}(z) + r^4\psi_k^{(2)}(z) + O(r^6). \tag{19}$$

**Remark 1.** The analytical dependencies of the complex potentials (18) and (19) on $r^2$ near $r = 0$ follows from the compactness of the operators defined by the system of functional Equations (12) and (13) in a Banach space and by the uniform convergence of successive approximations for small $r$ (Mityushev, 2000; Mityushev and Rogosin, 1999).

**Remark 2.** In Section 4, we introduce a dimensional rectangle of the unit area. This justifies the consideration of the formally small parameter $r^2$ through the ratio of the disk area $\pi r^2$ to the rectangle area.

The functions $\varphi_k^{(s)}$ and $\psi_k^{(s)}$ $(s = 0, 1, 2)$ in each inclusion are presented by their Taylor series. It is sufficient to take only first three terms

$$\varphi_k^{(s)}(z) = \alpha_{k,0}^{(s)} + \alpha_{k,1}^{(s)}(z-a_k) + \alpha_{k,2}^{(s)}(z-a_k)^2 + O((z-a_k)^3), \tag{20}$$

$$\psi_k^{(s)}(z) = \beta_{k,0}^{(s)} + \beta_{k,1}^{(s)}(z-a_k) + \beta_{k,2}^{(s)}(z-a_k)^2 + O\big((z-a_k)^3\big). \tag{21}$$

The precision $O((z-a_k)^3)$ is taken here by the reason explained below after Eqs. (22) and (24).

Introduce the auxiliary constants $\gamma_{m,l}^{(s)}$ for shortness

$$\gamma_{m,l}^{(s)} = (l+2)r^2\alpha_{m,l+2}^{(s)} + \overline{a_m}(l+1)\alpha_{m,l+1}^{(s)} + \beta_{m,l}^{(s)} \quad s = 0, 1;\ l = 1, 2. \tag{22}$$

Substitution of (18) and (19) in (12) and (13) yields

$$\begin{aligned}
&\left(\frac{\mu_1}{\mu} + \kappa_1\right) \sum_{p=0,1,2} r^{2p}\varphi_k^{(p)}(z) \\
&= \left(\frac{\mu_1}{\mu} - 1\right) \sum_{m\neq k} \sum_{l+s\leq 2} r^{2(s+l)}\overline{\gamma_{m,l}^{(s)}}(z-a_m)^{-l} \\
&+ \left(\frac{\mu_1}{\mu}(1+\kappa)B_0 - \left(\frac{\mu_1}{\mu} - 1\right) \sum_{s=0,1,2} r^{2s}\overline{\alpha_{k,1}^{(s)}}\right)(z-a_k) \\
&+ p_1 + O(r^6)
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\left(\kappa\frac{\mu_1}{\mu} + 1\right) \sum_{p=0,1,2} r^{2p}\left(\overline{a_k}\left(\varphi_k^{(p)}(z)\right)' + \psi_k^{(p)}(z)\right) \\
&= -\frac{\mu_1}{\mu} \sum_{s=0,1} r^{2(s+1)}\left(\varphi_k^{(s)}(z)\right)'(z-a_k)^{-1} \\
&+ \left(\kappa\frac{\mu_1}{\mu} - \kappa_1\right) \sum_{m\neq k} \sum_{l+s\leq 2} r^{2(l+s)}\overline{\alpha_{m,l}^{(s)}}(z-a_m)^{-l} \\
&- \left(1 - \frac{\mu}{\mu_1}\right) \sum_{m\neq k} r^4\overline{\gamma_{m,1}^{(0)}}(z-a_k)^{-1}(z-a_m)^{-2} \\
&+ \left(\frac{\mu_1}{\mu} - 1\right) \sum_{m\neq k} \sum_{l+s\leq 1} l r^{2(l+s+1)}\overline{\gamma_{m,l}^{(s)}}(z-a_m)^{-l-2} \\
&- \left(1 - \frac{\mu}{\mu_1}\right) \sum_{m\neq k} \sum_{l+s\leq 2} l r^{2(l+s)}\overline{\gamma_{m,l}^{(s)}}(\overline{a_k - a_m})(z-a_m)^{-l-1} \\
&+ \frac{\mu_1}{\mu}(1+\kappa)\Gamma_0(z-a_k) \\
&+ \sum_{m\neq k} r^2 q_m(z-a_m)^{-1} + q_0 + O(r^6),
\end{aligned} \tag{24}$$

where the sum $\sum_{l+s\leq 2}$ contains three terms with $l = 1, 2$ and $s = 0, 1$ satisfying the inequality $l + s \leq 2$. The sum $\Sigma_{m\neq k}$ means that $m$ runs over $\{1, 2, \ldots, n\}$ except $k$. Other sums are defined analogously. One can check that the higher order terms $(z-a_k)^l$ (for $l \geq 3$) in (20) and (21) produce terms of order $O(r^6)$ in (23) and (24). Such a rule takes place in general case when the terms $(z-a_k)^l$ yield $O(r^{2l})$. The constants $p_1$, $q_0$, $\alpha_{m,0}^{(s)}$ and $\beta_{m,0}^{(s)}$ $(m = 1, .., n$, $s = 0, 1, 2)$ in (23) and (24) should not be found since they determine parallel translations not important for the stress and deformation fields.

Introduce the following constants for shortness

$$\Omega_1 = \frac{\frac{\mu_1}{\mu} - 1}{(1+\kappa)\frac{\mu_1}{\mu}}, \quad \Omega_2 = \frac{\frac{\mu_1}{\mu} + \kappa_1}{(1+\kappa)\frac{\mu_1}{\mu}}, \quad \Omega_3 = \frac{\kappa\frac{\mu_1}{\mu} - \kappa_1}{(1+\kappa)\frac{\mu_1}{\mu}},$$
$$\Omega_4 = \frac{\kappa\frac{\mu_1}{\mu} + 1}{(1+\kappa)\frac{\mu_1}{\mu}}. \tag{25}$$

Selecting the terms with the same powers $r^{2p}$, we arrive at the following iteration scheme for Eqs. (23) and (24). The zero terms $(p = 0)$ are calculated by

$$\varphi_k^{(0)}(z) = B_0(\Omega_1 + \Omega_2)^{-1}(z-a_k), \tag{26}$$

$$\psi_k^{(0)}(z) = \Gamma_0\Omega_4^{-1}(z-a_k). \tag{27}$$

The higher order terms $\varphi_k^{(1,2)}(z)$ and $\psi_k^{(1,2)}(z)$ are written in Supplementary (see formulae (S1)–(S4)).

It follow from (12)–(16) that

$$\varphi(z) = r^2\varphi^{(1)}(z) + r^4\varphi^{(2)}(z) + O(r^6),$$

where

$$\varphi^{(1)}(z) = \Omega_1\Omega_4^{-1}\overline{\Gamma_0} \sum_{k=1}^{n} (z-a_k)^{-1}. \tag{28}$$

Analogously, the function $\psi$ has the form

$$\psi(z) = r^2\psi^{(1)}(z) + r^4\psi^{(2)}(z) + O(r^6),$$

where

$$\begin{aligned}
\psi^{(1)}(z) = \sum_{k=1}^{n} \Big(&2(\Omega_4 - \Omega_2)(\Omega_1 + \Omega_2)^{-1}B_0(z-a_k)^{-1} \\
&+ \Omega_1\Omega_4^{-1}\overline{\Gamma_0}\overline{a_k}(z-a_k)^{-2}\Big).
\end{aligned} \tag{29}$$

The second order terms $\varphi^{(2)}(z)$ and $\psi^{(2)}(z)$ are written in Supplementary (see formulae (S5) and (S6)).

The present computational scheme was implemented in *Mathematica*© to perform symbolic computations. It can be easily extended to higher orders. The main problem is to see the result since very long formulae arise after symbolic computations. We shall treat this problem in a separate paper.

## 4. Averaged fields in finite composites

The displacement $(u, v)$ are calculated by formulae (Muskhelishvili, 1966)

$$u + iv = \begin{cases} \frac{1}{2\mu_1}\left(\kappa_1\varphi_k(t) - t\overline{\varphi_k'(t)} - \overline{\psi_k(t)}\right), & z \in D_k, \\ \frac{1}{2\mu}\left(\kappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}\right), & z \in D. \end{cases} \tag{30}$$

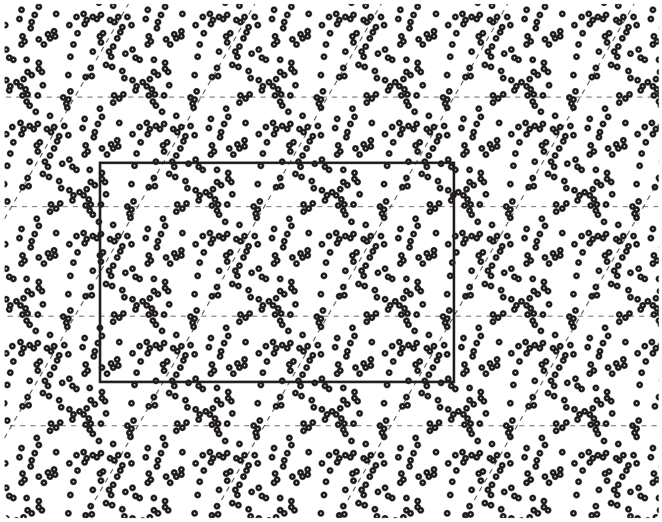

**Fig. 1.** Schematic presentation of the infinite number of disks on the plane with the rectangle $Q_n$ containing a finite number of disks. In Sections 2 and 3, we consider the problem when inclusions exterior to $Q_n$ are absent. In Section 4, we consider the infinite set of points on the plane and pass to the limit when $Q_n$ extends and embraces the corresponding points, i.e., the boundary of $Q_n$ tends to the infinite point, as $n \to \infty$.

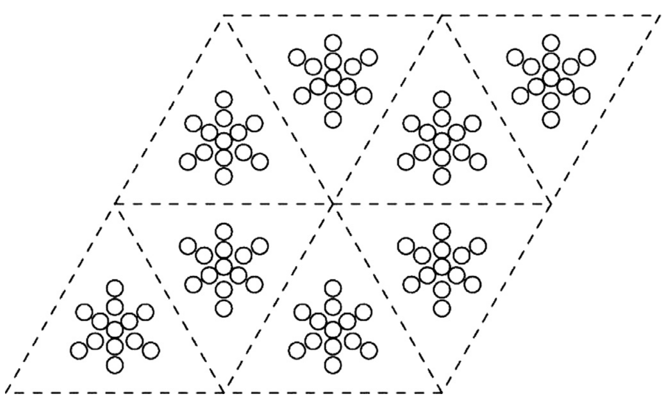

**Fig. 2.** The hexagonal cell with 104 inclusions symmetrically generated by 13 inclusions in marked triangle. The coordinates of 13 inclusions are given by (S13) in Supplementary.

Components of the strain tensor are given by

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad 2\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \tag{31}$$

Let $n$ circular inclusions lie in the rectangle $Q_n = \{x + iy \in \mathbb{C} : -\frac{a}{2} < x < \frac{a}{2}, -\frac{b}{2} < y < \frac{b}{2}\}$ (see Fig. 1) whose area $|Q_n| = ab$.

We define the macroscopic bulk and shear moduli of this rectangle as follows

$$\mu_e^{(n)} = \frac{\langle \sigma_{xx} - \sigma_{yy} \rangle_n}{2\langle \epsilon_{xx} - \epsilon_{yy} \rangle_n}, \tag{32}$$

$$k_e^{(n)} = \frac{\langle \sigma_{xx} + \sigma_{yy} \rangle_n}{2\langle \epsilon_{xx} + \epsilon_{yy} \rangle_n}, \tag{33}$$

where the average $\langle \cdot \rangle_n = \frac{1}{|Q_n|} \iint\limits_{Q_n} \cdot dxdy$. Denote by $D_0$ the complement of the closed disks $D_k$ to the rectangle $Q_n$.

**Lemma 1.** *Let a function $g(z)$ be analytic in $Q_n'$ the complement of $Q_n \cup \partial Q_n$ to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then,*

$$\lim_{\substack{a\to\infty \\ b\to\infty}} \frac{1}{ab} \int_{\partial Q_n} g(t)\, d\bar{t} = 0 \tag{34}$$

**Proof.** It follows from the estimation

$$\left| \frac{1}{ab} \int_{\partial Q_n} g(t)\, d\bar{t} \right| \le \max_{t \in \partial Q_n} |g(t)| \frac{2(a+b)}{ab} \tag{35}$$

and boundness of $|g(z)|$ in $Q_n'$. □

The deformations and stresses are related to the complex potentials by formulae (Muskhelishvili, 1966)

$$\epsilon_{xx} + \epsilon_{yy} = \begin{cases} \frac{\kappa_1 - 1}{\mu_1} \mathrm{Re}\, \varphi_k'(z), & z \in D_k, \\ \frac{\kappa - 1}{\mu} \mathrm{Re}\, \varphi_0'(z), & z \in D, \end{cases} \tag{36}$$

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\mathrm{Re}\, \varphi_k'(z), & z \in D_k, \\ 4\mathrm{Re}\, \varphi_0'(z), & z \in D, \end{cases} \tag{37}$$

$$\epsilon_{xx} - \epsilon_{yy} = \begin{cases} -\frac{1}{\mu_1} \mathrm{Re}\left(\bar{z}\varphi_k''(z) + \psi_k'(z)\right), & z \in D_k, \\ -\frac{1}{\mu} \mathrm{Re}\left(\bar{z}\varphi_0''(z) + \psi_0'(z)\right), & z \in D, \end{cases} \tag{38}$$

$$\sigma_{xx} - \sigma_{yy} = \begin{cases} -2\mathrm{Re}\left(\bar{z}\varphi_k''(z) + \psi_k'(z)\right), & z \in D_k, \\ -2\mathrm{Re}\left(\bar{z}\varphi_0''(z) + \psi_0'(z)\right), & z \in D. \end{cases} \tag{39}$$

Calculate the integral

$$\frac{1}{|Q|} \left( \iint_{D_0} \frac{\partial}{\partial z} (B_0 z)\, dxdy \right) = B_0 \left( 1 - \frac{n\pi r^2}{|Q|} \right). \tag{40}$$

We shall apply Green's formula in complex form to calculate the integrals from (32) and (33)

$$\int_D \frac{\partial w(z)}{\partial z} dxdy = -\frac{1}{2i} \int_{\partial D} w(t) d\bar{t}. \tag{41}$$

The differential $d\bar{t}$ on each circle can be transformed as follows

$$d\bar{t} = d\left( \frac{r^2}{t - a_k} + \overline{a_k} \right) = -\left( \frac{r}{t - a_k} \right)^2 dt, \quad |t - a_k| = r. \tag{42}$$

Using (40)–(42) we get

$$\langle \sigma_{xx} + \sigma_{yy} \rangle_n = 4\mathrm{Re} \left\{ B_0 \left( 1 - \frac{n\pi r^2}{|Q|} \right) - \frac{1}{2i|Q|} \times \sum_{k=1}^{n} \int_{\partial D_k} \left[ (\varphi(z) - \varphi_k(z)) \left( \frac{r}{z - a_k} \right)^2 \right] dz \right\} + O_n \tag{43}$$

and

$$\langle \epsilon_{xx} + \epsilon_{yy} \rangle_n = \mathrm{Re} \left\{ \frac{\kappa - 1}{\mu} B_0 \left( 1 - \frac{n\pi r^2}{|Q|} \right) - \frac{1}{2i|Q|} \sum_{k=1}^{n} \int_{\partial D_k} \times \left[ \left( \frac{\kappa - 1}{\mu} \varphi(z) - \frac{\kappa_1 - 1}{\mu_1} \varphi_k(z) \right) \left( \frac{r}{z - a_k} \right)^2 \right] dz \right\} + O_n. \tag{44}$$

Here, the integrals over $\partial Q_n$ vanishing in the limit $n \to \infty$ (see Lemma 1) are denoted by $O_n$ for shortness.

Along similar lines we calculate $\langle \sigma_{xx} - \sigma_{yy} \rangle_n$ and $\langle \epsilon_{xx} - \epsilon_{yy} \rangle_n$ (see formulae (S7), (S8) in Supplementary). After substitution of (18) and (19) and (26)–(29) into the integrals (43) and (44) and (S7) and (S8) these integral can be easily calculated by residues. The ultimate results hold the accuracy $O(r^6)$. The explicit formulae (S9)–(S12) are written in Supplementary.

## 5. Passage to composites represented by RVE

### 5.1. General

The asymptotic analytical formulae for the averaged stresses and deformations are written in the previous section for a finite number of inclusions $n$ in the plane. Direct computation of the effective constants by these formulae gives the effective properties of clusters (Mityushev and Adler, 2002; Mityushev and Rylko, 2013) consisting of $n$ inclusions diluted in the plane. These formulae hold for any number $n$ given in symbolic form. Therefore, one can pass to the limit $n \to \infty$ but this passage must be properly justified.

First, consider the pure geometrical sums arisen in the above formulae

$$e_2(n) = \frac{1}{n^2}\sum_{k=1}^{n}\sum_{m\neq k}\frac{1}{(a_k-a_m)^2}, \tag{45}$$

$$e_3^{(1)}(n) = \frac{1}{n^2}\sum_{k=1}^{n}\sum_{m\neq k}\frac{\overline{a_k-a_m}}{(a_k-a_m)^3}, \tag{46}$$

$$e_{22}(n) = \frac{1}{n^3}\sum_{k=1}^{n}\sum_{m\neq k}\sum_{l\neq m}\frac{1}{(a_k-a_m)^2}\,\frac{1}{\overline{(a_m-a_l)^2}}, \tag{47}$$

$$\widetilde{e}_{22}(n) = \frac{1}{n^3}\sum_{k=1}^{n}\sum_{m\neq k}\sum_{l\neq m}\frac{1}{(a_k-a_m)^2}\,\frac{1}{(a_m-a_l)^2}, \tag{48}$$

$$e_{33}^{(1)}(n) = \frac{1}{n^3}\sum_{k=1}^{n}\sum_{m\neq k}\sum_{l\neq m}\frac{\overline{a_k-a_m}}{(a_k-a_m)^3}\,\frac{a_m-a_l}{\overline{(a_m-a_l)^3}}. \tag{49}$$

Consider infinite number of the mutually disjoint disks $D_k = \{z \in \mathbb{C} : |z-a_k| < r\}$ $(k = 1, 2, \ldots)$ on the complex plane. Let the centers $a_k$ are ordered in such a way that

$$|a_1| \le |a_2| \le |a_3| \le \ldots. \tag{50}$$

Let $D$ be the complement of the closed disks $|z-a_k| \le r$ $(k = 1, 2, \ldots)$ to the complex plane. As above let $Q_n$ denote a rectangle containing first $n$ disks $D_1$, $D_2$, ..., $D_n$; $F_n = Q_n \setminus \cup_{k=1}^{n}(D_k \cup \partial D_k)$ and $|Q_n|$ its area (see Fig. 1). Let the finitely connected domains $F_n$ tend to $D$ as $n \to \infty$, i.e., $\partial Q_n$ tends to the infinite point. The concentration of inclusions is introduced as the limit[1]

$$f = \lim_{n\to\infty}\frac{n\pi r^2}{|Q_n|} = N\pi r^2, \tag{51}$$

where $N$ is the average number of inclusions per unit area, i.e., $|Q_n| \sim \frac{n}{N}$ as $n \to \infty$. This is equivalent to introduction of a dimensionless length scale.

The conditional convergence of (45) and (46), as $n \to \infty$, is the principal difficulty to constructively apply the MMM principle by Hashin and to deduce analytical formulae for the effective constants. The limit (45) was first theoretically investigated in Mityushev (1999) where the question of the conditional convergence was resolved and the results were applied to calculation of the effective conductivity.

It follows from the homogenization theory outlined in Section 2 that it is not necessary to study general structures with infinite number of inclusions on the plane as it was done in Mityushev (1999). Because investigation of the statistically homogeneous composites can be reduced to investigation of the RVE. We follow these lines in the next sections and consider a doubly periodic composite with a finite number of inclusion $N$ per periodicity cell. The area of the periodicity cell is normalized to unity that is agree with (51). This number $N$ is kept in symbolic form that gives the possibility to apply an arbitrary probabilistic distribution of $N$ disks in the representative cell.

[1] It is assumed that this limit exists.

### 5.2. The limit of $e_2(n)$

Introduce the limit of (45)

$$e_2 := \lim_{n\to\infty} e_2(n) = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sum_{m\neq k}\frac{1}{(a_k-a_m)^2}. \tag{52}$$

In the next Section 5.2.1, we investigate this limit for regular composites following Rayleigh (1892) by use of the Eisenstein summation (Weil, 1976). In Section 5.2.2, we investigate the limit (52) for statistically homogeneous composites reduced to doubly periodic ones.

#### 5.2.1. Regular composites

In the partial case, when the centers $a_k$ form a regular lattice, formula (52) can be treated as Rayleigh's formula (Rayleigh, 1892). For instance, let the centers $a_k$ generate the square array $Q = \left\{\frac{p}{\sqrt{N}} + \frac{iq}{\sqrt{N}} : p, q \in \mathbb{Z}\right\}$. Then, the average number of disks per cell $N = 1$ and (52) becomes

$$e_2 = \sum_{p,q\in\mathbb{Z}\setminus\{0\}}^{e}\frac{1}{(p+iq)^2}. \tag{53}$$

The series (53) is conditionally convergent, hence, its value depends on the order of summation. Here, the order is fixed by using of the Eisenstein summation (Weil, 1976)

$$\sum_{p,q}^{e} := \lim_{M_2\to+\infty}\lim_{M_1\to+\infty}\sum_{q=-M_2}^{M_2}\sum_{p=-M_1}^{M_1}. \tag{54}$$

The iterated limit in (54) means that the rectangle of summation shown in Fig. 1, first, is extended along the $x$-axis and after along the $y$-axis.

The case $N = 1$ yields the famous Eisenstein–Rayleigh lattice sum

$$S_2 = \sum_{p,q\in\mathbb{Z}\setminus\{0\}}^{e}\frac{1}{(p+iq)^2}. \tag{55}$$

Rayleigh (1892) applied the Eisenstein summation (54) having considered the conductivity problem for the square lattice[2] and found $S_2 = \pi$. The inner limit in the $x$-direction in (54) corresponds to the direction of external flux. A physical justification of (54) based on the polarization effect was presented in McPhedran and McKenzie (1978) and Perrins et al. (1979). The complete mathematical investigation of the Eisenstein-Rayleigh with different orders of summations can be found in Mityushev (1997a).

#### 5.2.2. Statistically homogeneous composites

As it was noted at the end of Section 5.1 the passage to the limit in (52) for statistically homogeneous composites is reduced to doubly periodic composites. In order to describe such a composite and further to deduce constructive formulae we consider a lattice $\mathcal{G}$ which is defined by two translation vectors expressed by complex numbers $\omega_1$, $\omega_2$. Without loss of generality we assume that $\omega_1 > 0$ and $\mathrm{Im}\,\frac{\omega_2}{\omega_1} > 0$. Introduce the zero-th cell

$$\mathcal{G}_{(0,0)} := \left\{z = t_1\omega_1 + t_2\omega_2 \in \mathbb{C} : -\frac{1}{2} < t_1, t_2 < \frac{1}{2}\right\}.$$

Let the area of $\mathcal{G}_{(0,0)}$ be normalized to unity, hence

$$\omega_1 \mathrm{Im}\,\omega_2 = 1. \tag{56}$$

[2] Rayleigh (1892) did not cite Eisenstein's result (1847) and addressed to Weierstrass' investigations (1856). Perhaps, it is related to that Eisenstein treated formally his series without uniform convergence (Weil, 1976).

Introduce the numbers $P = p\omega_1 + q\omega_2$ for integer $p$ and $q$ and the $P$-cell

$$\mathcal{G}_{(p,q)} := \mathcal{G}_{(0,0)} + P = \left\{z \in \mathbb{C} : z - P \in \mathcal{G}_{(0,0)}\right\}.$$

Let $\mathcal{D}_k$ denote mutually disjoint disks $|z - a_k| < r$ $(k = 1, 2, \ldots, N)$ whose centers $a_k$ are located in the zeroth cell $\mathcal{G}_{(0,0)}$. The location of inclusions in other cells periodically repeats the location in $\mathcal{G}_{(0,0)}$ in the torus topology. The normalization (56) is consistent with the definition of concentration (51). Introduce the multiply connected domain $\mathcal{D}_0 = \mathcal{G}_{(0,0)} \setminus \bigcup_{k=1}^{N}(\mathcal{D}_k \cup \partial\mathcal{D}_k)$ obtained by removing of the inclusions from the zero-th cell. We say that the set of centers

$$A = \{a_k + P, k = 1, 2, \ldots, N; P = p\omega_1 + q\omega_2,\ \ p, q \in \mathbb{Z}\} \tag{57}$$

generate a double periodic structure. This set $A$ can be reordered in accordance with (50). It is worth noting that the set $A$ in general does not form a lattice, since the points $a_k$ $(k = 1, 2, \ldots, N)$ satisfy only the non-overlapping condition $|a_k - a_m| \geq 2r$ $(m \neq k)$ and arbitrarily located in the zero cell. The number $N$ can be taken arbitrarily large what gives a possibility to consider the set $\{a_k, k = 1, 2, \ldots, N\}$ as a statistically random set of points satisfying any desired probabilistic non-overlapping distribution.

Let the doubly periodic host domain $\mathcal{D}_0 + P$ be occupied by an isotropic elastic material with the shear modulus $\mu$ and the bulk modulus $k$. Let the inclusions $\mathcal{D}_k + P$ be occupied by an isotropic elastic material characterized by the shear modulus $\mu_1$ and the bulk modulus $k_1$.

It was shown in Mityushev (1999), Mityushev and Rylko (2012) and Mityushev and Rylko (2013) that for macroscopically (ideally isotropic) composites

$$e_2 = \pi. \tag{58}$$

This formula generalizes Rayleigh's equation $S_2 = \pi$ to random composites.

### 5.3. The limit of $e_3^{(1)}(n)$

We now proceed to investigate the limit formula (46) for macroscopically isotropic composites represented by doubly periodic structures. Let a structure is determined by the centers set (57). First, consider the conditionally lattice sum

$$S_3^{(1)} = \sum_{P\neq 0}^{e} \frac{\bar{P}}{P^3}, \tag{59}$$

where $\sum_{P\neq 0}^{e}$ stands for the Eisenstein summation over the integer numbers $p$ and $q$ except the term $p = q = 0$. Consider the function introduced by Natanzon (1935) and developed by Grigolyuk and Fil'shtinskij (1992)[3]

$$\wp_1'(z) = -2\sum_{P\neq 0}\left(\frac{\overline{P}}{(z-P)^3} + \frac{\overline{P}}{P^3}\right), \tag{60}$$

where we follow the notations (Grigolyuk and Fil'shtinskij, 1992). The series (60) is absolutely convergent. Using the Eisenstein summation we introduce the function

$$E_3^{(1)}(z) = \sum_{P}^{e} \frac{\overline{z-P}}{(z-P)^3}. \tag{61}$$

The functions (60) and (61) are related by equation

$$E_3^{(1)}(z) = -\frac{1}{2}\bar{z}\wp'(z) + \frac{1}{2}\wp_1'(z) + S_3^{(1)}, \tag{62}$$

[3] See the review Filshtinsky and Mityushev (2014) with an extended list of references.

where $\wp(z)$ denotes the Weierstrass elliptic function (Weil, 1976). Grigolyuk and Fil'shtinskij (1992) expressed Natanzon's function $\wp_1'(z)$ in terms of elliptic functions

$$\begin{aligned}\pi\wp_1'(z) = {} & \frac{1}{3}\wp''(z) + [\zeta(z) - (S_2 - \pi)z]\wp'(z) \\ & - 2(S_2 - \pi)\wp(z) - 10S_4,\end{aligned} \tag{63}$$

where the following lattice sum are used

$$S_2 = \sum_{P\neq 0}\frac{1}{P^2}, \quad S_4 = \sum_{P\neq 0}\frac{1}{P^4}. \tag{64}$$

Formula (63) is obtained from Filshtinsky's formulae (15) and (25) from Appendix 2 of Grigolyuk and Fil'shtinskij (1992) by using of (56), Legendre's identity and formula $S_2 = \frac{2}{\omega_1}\zeta\left(\frac{\omega_1}{2}\right)$ deduced in Mityushev (1997b). Substitution of (63) and (64) into (62) yields

$$\begin{aligned}E_3^{(1)}(z) = {} & -\frac{1}{2}\bar{z}\wp'(z) + \frac{1}{6\pi}\wp''(z) + \frac{1}{2}\left[\frac{\zeta(z)}{\pi} - \left(\frac{S_2}{\pi} - 1\right)z\right]\wp'(z) \\ & - \left(\frac{S_2}{\pi} - 1\right)\wp(z) - \frac{5}{\pi}S_4 + S_3^{(1)}.\end{aligned} \tag{65}$$

The latter formula is simplified for the hexagonal cell when $S_2 = \pi$ and $S_4 = 0$ (Akhiezer, 1990)

$$E_3^{(1)}(z) = -\frac{1}{2}\bar{z}\wp'(z) + \frac{1}{6\pi}\wp''(z) + \frac{1}{2\pi}\zeta(z)\wp'(z) + S_3^{(1)}. \tag{66}$$

Introduce the designation

$$\langle F(z)\rangle = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{m=1}^{N}F(a_k - a_m), \tag{67}$$

The limit for doubly periodic structures can be calculated by formula

$$e_3^{(1)} = \lim_{n\to\infty} e_3^{(1)}(n) = \langle E_3^{(1)}(z)\rangle, \tag{68}$$

where it is assumed that $E_3^{(1)}(0) := S_3^{(1)}$. It was proved in Yakubovich and Mityushev (2016) that $S_3^{(1)} = \frac{\pi}{2}$ for the hexagonal lattice.

### 5.4. The limit of $e_{33}^{(1)}(n)$

Let $k$ be a natural number and $j = 0, 1$. Introduce the functions

$$E_k^{(j)}(z) = \sum_{P}^{e}\frac{\left(\overline{z-P}\right)^j}{(z-P)^k}. \tag{69}$$

The superscript $j$ for $j = 0$ will be omitted below, i.e., we write $E_k^{(0)}(z) = E_k(z)$ for shortness. Calculate the limit for the doubly periodic structures

$$e_{22} = \lim_{n\to\infty} e_{22}(n) = \frac{1}{N^3}\sum_{k=1}^{N}\sum_{m=1}^{N}\sum_{l=1}^{N}E_2(a_k - a_m)\overline{E_2(a_m - a_l)} \tag{70}$$

Analogously

$$\tilde{e}_{22} = \lim_{n\to\infty}\tilde{e}_{22}(n) = \frac{1}{N^3}\sum_{k=1}^{N}\sum_{m=1}^{N}\sum_{l=1}^{N}E_2(a_k - a_m)E_2(a_m - a_l) \tag{71}$$

and

$$e_{33}^{(1)} = \lim_{n\to\infty} e_{33}^{(1)}(n) = \frac{1}{N^3}\sum_{k=1}^{N}\sum_{m=1}^{N}\sum_{l=1}^{N}E_3^{(1)}(a_k - a_m)\overline{E_3^{(1)}(a_m - a_l)}. \tag{72}$$

## 6. Effective constants

For simplicity, we consider a macroscopically isotropic material described by two effective constants $\mu_e$ and $k_e$.

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### 6.1. Effective constants up to O(f³)

Consider the limit relation (32) with (S11) and (S12) up to $O(f^3)$ terms from Supplementary, as $n$ tends to infinity. We have

$$\frac{\mu_e}{\mu} = 1 + (1+\kappa)M_1 f + (1+\kappa)M_2 f^2 + O(f^3), \tag{73}$$

where

$$M_1 = \frac{\Omega_1}{\Omega_4} = \frac{\frac{\mu_1}{\mu} - 1}{\kappa\frac{\mu_1}{\mu} + 1}, \tag{74}$$

$$M_2 = M_1^2\left(\kappa + \frac{2e_3^{(1)}}{\pi}\frac{\overline{\Gamma_0}}{\Gamma_0}\right) + 2M_1K_1\frac{B_0}{\Gamma_0}\frac{e_2}{\pi}, \tag{75}$$

$$K_1 = \frac{(\kappa-1)\frac{\mu_1}{\mu} - (\kappa_1 - 1)}{\kappa_1 - 1 + 2\frac{\mu_1}{\mu}}. \tag{76}$$

The limit of (33) with (S9) and (S10) from Supplementary yields

$$\frac{k_e}{k} = 1 + (\kappa+1)\frac{K_1}{\kappa-1}f + (\kappa+1)K_2 f^2 + O(f^3), \tag{77}$$

where

$$K_2 = \frac{K_1}{\kappa-1}\left[2\left(\frac{K_1}{\kappa-1}\right) + M_1\frac{\kappa_1 + \frac{\mu_1}{\mu}}{1+\kappa_1}\frac{\overline{\Gamma_0}}{B_0}\frac{e_2}{\pi} + M_1\frac{1-\frac{\mu_1}{\mu}}{1+\kappa_1}\frac{\Gamma_0}{B_0}\frac{\overline{e_2}}{\pi}\right]. \tag{78}$$

Using the relation $k = \frac{2\mu}{\kappa-1}$ we arrive at the same first order coefficients $M_1$ and $K_1$ as in (1).

The coefficients $M_2$ and $K_2$ formally depend on $B_0$ and $\Gamma_0$ which are expressed in terms of the stresses at infinity by formulae (8). Moreover, $M_2$ and $K_2$ formally depend on the complex values $e_3^{(1)}$ and $e_{22}$. However, the homogenization theory implies that the coefficients $M_2$ and $K_2$ must be real numbers and do not depend on the choice of $B_0$ and $\Gamma_0$. This illusory contradiction is based on the conditional convergence of the sums $e_2$ and $e_3^{(1)}$. Consider the sum $e_3^{(1)}$ defined by (68) and (66). It follows from (66) that the conditionally convergent part of $e_3^{(1)}$ is the sum $S_3^{(1)}$ defined by (59) where the Eisenstein summation is used.

We shall check that $\frac{B_0}{\Gamma_0}S_2$ and $\frac{\overline{\Gamma_0}}{\Gamma_0}S_3^{(1)}$ are invariant under change of $B_0$ and $\Gamma_0$. The invariance for $S_2$ was proved in Mityushev (1997a) in terms of conductivity. We now demonstrate the invariance for few stress states at infinity. The complete proof is long and follows the lines of Mityushev (1997a). Compare two states (1) $\sigma_{xx}^\infty = 1$, $\sigma_{yy}^\infty = \sigma_{xy}^\infty = 0$ and (2) $\sigma_{yy}^\infty = 1$, $\sigma_{xx}^\infty = \sigma_{xy}^\infty = 0$. The second state can be obtained from the first one by rotation about the angle $\frac{\pi}{2}$. Let $\omega_1^{(1)}$ and $\omega_2^{(1)}$ be two fundamental vectors. It is convenient to compare these states through rotation of the fundamental vectors $\omega_1^{(2)} = \omega_1^{(1)}\exp(\frac{i\pi}{2})$ and $\omega_2^{(2)} = \omega_2^{(1)}\exp(\frac{i\pi}{2})$ where the superscript corresponds to the number of states. It follows from (64) with $P = p\omega_1^{(j)} + q\omega_2^{(j)}$ $(j = 1, 2)$ that

$$S_2(\omega_1^{(2)}, \omega_2^{(2)}) = [\omega_1^{(2)}]^{-2}S_2(1,\tau) = -S_2(\omega_1^{(1)}, \omega_2^{(1)})$$

and from (59) that

$$S_3^{(1)}(\omega_1^{(2)}, \omega_2^{(2)}) = \overline{\omega_1^{(2)}}[\omega_1^{(2)}]^{-3}S_3^{(1)}(1,\tau) = S_3^{(1)}(\omega_1^{(1)}, \omega_2^{(1)}),$$

where $\tau = \frac{\omega_2^{(1)}}{\omega_1^{(1)}} = \frac{\omega_2^{(2)}}{\omega_1^{(2)}}$. One can see that

$$\frac{B_0^{(2)}}{\Gamma_0^{(2)}}S_2(\omega_1^{(2)}, \omega_2^{(2)}) = \frac{B_0^{(1)}}{\Gamma_0^{(1)}}S_2(\omega_1^{(1)}, \omega_2^{(1)}) \tag{79}$$

and

$$\frac{\overline{\Gamma_0^{(2)}}}{\Gamma_0^{(2)}}S_3^{(1)}(\omega_1^{(2)}, \omega_2^{(2)}) = \frac{\overline{\Gamma_0^{(1)}}}{\Gamma_0^{(1)}}S_3^{(1)}(\omega_1^{(1)}, \omega_2^{(1)}). \tag{80}$$

Consider now the stress states 3) $\sigma_{xx}^\infty = \sigma_{yy}^\infty = 0$, $\sigma_{xy}^\infty = 1$, i.e., $B_0 = 0$, $\Gamma_0 = i$ and 4) $\sigma_{yy}^\infty = 1$, $\sigma_{xx}^\infty = -1$, $\sigma_{xy}^\infty = 0$, i.e., $B_0 = 0$, $\Gamma_0 = 1$. The state 4) is obtained from 3) by rotation about the angle $-\frac{\pi}{4}$. Then, the relation (79) is trivial (equivalent to $0 = 0$) and

$$S_3^{(1)}(\omega_1^{(2)}, \omega_2^{(2)}) = -S_3^{(1)}(\omega_1^{(1)}, \omega_2^{(1)}).$$

One can check that the relations (80) is fulfilled also in this case. Other stress states can be obtained by linear combinations of the considered four states.

This result enable us to take any pair $B_0$ and $\Gamma_0$. We will take $B_0 = 0$, $\Gamma_0 = i$ and $B_0 = 1$, $\Gamma_0 = 0$ for convenience.

### 6.2. Effective constants up to O(f⁴)

In the present section, we consider the hexagonal representative cell when $\omega_1 = \sqrt[4]{\frac{4}{3}}$ and $\omega_2 = \sqrt[4]{\frac{4}{3}}\,e^{\frac{\pi i}{3}}$ and the considered random structure is macroscopically isotropic. Then, $e_2 = \pi$ and $e_3^{(1)} = \frac{\pi}{2}$.

Let $B_0 = 0$, $\Gamma_0 = i$. Then, the coefficient (75) becomes

$$M_2 = M_1^2(\kappa - 1). \tag{81}$$

The next approximation yields

$$\frac{\mu_e}{\mu} = 1 + (1+\kappa)M_1 f + (1+\kappa)M_2 f^2 + (1+\kappa)M_3 f^3 + O(f^4), \tag{82}$$

where

$$M_3 = M_1^2\left(\kappa^2 M_1 - 2\kappa M_1 + 6\frac{e_4}{\pi^2} - K_1\frac{\widetilde{e}_{22}}{\pi^2} + 4M_1\frac{e_{33}^{(1)}}{\pi^2} + K_1\frac{e_{22}}{\pi^2}\right) \tag{83}$$

Let $B_0 = 1$, $\Gamma_0 = 0$. Then, the coefficient (78) becomes

$$K_2 = 2\left(\frac{K_1}{\kappa-1}\right)^2$$

and

$$\frac{k_e}{k} = 1 + \frac{\kappa+1}{\kappa-1}K_1 f + (\kappa+1)K_2 f^2 + (\kappa+1)K_3 f^3 + O(f^4), \tag{84}$$

where

$$K_3 = 4\left(\frac{K_1}{\kappa-1}\right)^3 + 2\frac{M_1K_1^2}{\kappa-1}\frac{e_{22}}{\pi^2}. \tag{85}$$

One can see that $\mu_e$ and $k_e$ do not depend on locations of inclusions up to $O(f^3)$ and the coefficients $M_3$ and $K_3$ do depend.

Consider two numerical examples.

1. Take 13 disks of radius $r = 0.022$ located at the basic triangle as displayed in Fig. 2. Their numerical coordinates are written in Supplementary (see formula (S13)). Other triangles are obtained by symmetries with respect to the sides of the basic and generated triangles. As a result we obtain the hexagonal periodicity cell containing 104 disks. The constructed structure has three lines of symmetry (dashed lines in Fig. 2). This implies that it is macroscopically isotropic. The $e$-sums of the considered structure take the values: $e_2 = \pi$, $e_4 = 0$, $e_3^{(1)} = \frac{\pi}{2}$, $e_{22} = 55.1009$, $\widetilde{e}_{22} = 9.869604$, $e_{33}^{(1)} = 60.9421$. Their substitution into (82) and

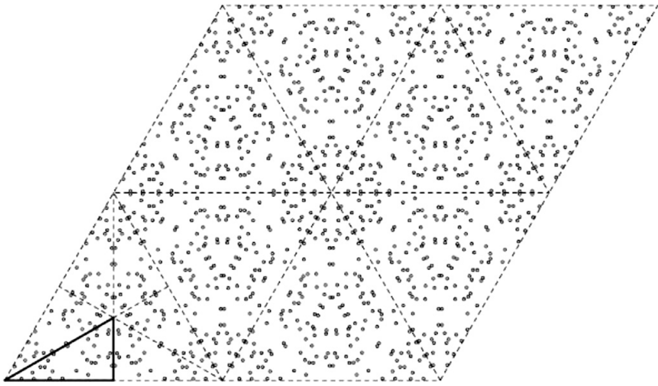

**Fig. 3.** The hexagonal cell with 1296 inclusions symmetrically generated by 27 inclusions in marked triangle. The coordinates of 27 inclusions are given by (S14) in Supplementary.

(84) yields up to $O(f^4)$

$$\frac{\mu_e}{\mu} = 1 + \frac{2\mu(k+\mu)(\mu_1-\mu)}{2\mu\mu_1 + k(\mu+\mu_1)} f + \frac{4\mu^2(k+\mu)(\mu-\mu_1)^2}{2\mu\mu_1 + k(\mu+\mu_1)^2} f^2$$

$$+2\mu\left(\frac{\mu}{k}+1\right)$$

$$\left(\frac{5.58289(\mu-\mu_1)^2\left(\mu_1\left(\frac{2\mu}{k}+1\right) - \frac{2\mu\mu_1}{k_1} - \mu_1\right)}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^2\left(\frac{2\mu\mu_1}{k_1}+2\mu_1\right)}\right.$$

$$+\frac{(\mu-\mu_1)^2\left(-\mu_1\left(\frac{2\mu}{k}+1\right)+\frac{2\mu\mu_1}{k_1}+\mu_1\right)}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^2\left(\frac{2\mu\mu_1}{k_1}+2\mu_1\right)}$$

$$-\frac{\left(\frac{2\mu}{k}+1\right)^2(\mu-\mu_1)^3}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^3} + \frac{2\left(\frac{2\mu}{k}+1\right)(\mu-\mu_1)^3}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^3}$$

$$\left.-\frac{24.6989(\mu-\mu_1)^3}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^3}\right) f^3. \qquad (86)$$

2. In order to get a macroscopically isotropic composite take 27 disks of radius $r = 0.003642$ randomly located at the basic triangle (see Fig. 3)[4]. One of the statistical realization is shown in Fig. 3. Their numerical coordinates are written in Supplementary (see formula (S14)). Other 47 triangles are obtained by symmetries with respect to the sides of the basic and generated triangles. As a result we obtain the hexagonal periodicity cell containing 1296 disks. The constructed structure has three lines of symmetry (dashed lines in Fig. 3). This implies that it is macroscopically isotropic. The $e$-sums of the considered structure take the values: $e_2 = \pi$, $e_4 = 0$, $e_3^{(1)} = \frac{\pi}{2}$, $e_{22} = 83.393648$, $\tilde{e}_{22} = 9.869604$, $e_{33}^{(1)} = 79.375512$. Their substitution into (82) and (84) yields up to $O(f^4)$

$$\frac{\mu_e}{\mu} = 1 + \frac{2\mu(k+\mu)(\mu_1-\mu)}{2\mu\mu_1 + k(\mu+\mu_1)} f + \frac{4\mu^2(k+\mu)(\mu-\mu_1)^2}{2\mu\mu_1 + k(\mu+\mu_1)^2} f^2$$

$$+2\mu\left(\frac{\mu}{k}+1\right)$$

$$\times\left(\frac{8.44954(\mu-\mu_1)^2\left(\mu_1\left(\frac{2\mu}{k}+1\right) - \frac{2\mu\mu_1}{k_1} - \mu_1\right)}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^2\left(\frac{2\mu\mu_1}{k_1}+2\mu_1\right)}\right.$$

$$+\frac{(\mu-\mu_1)^2\left(-\mu_1\left(\frac{2\mu}{k}+1\right)+\frac{2\mu\mu_1}{k_1}+\mu_1\right)}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^2\left(\frac{2\mu\mu_1}{k_1}+2\mu_1\right)}$$

[4] Here, randomly means the uniform non-overlapping distribution in the considered triangle

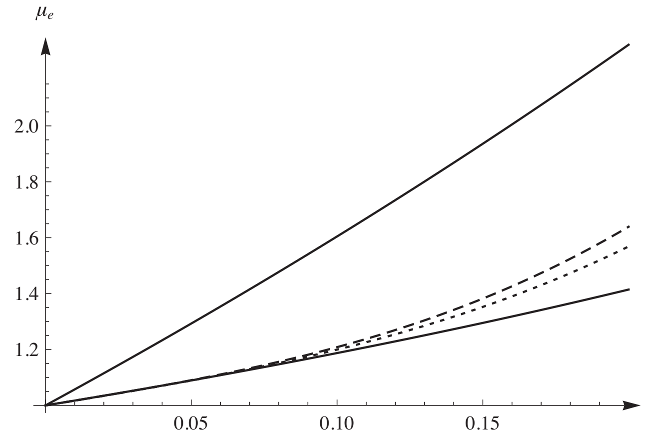


**Fig. 4.** The dependencies of $\frac{\mu_e}{\mu}$ on the concentration $f$ for the composites presented in Figs 2 and 3 when $\frac{\mu_1}{\mu} = 15$, $\frac{k_1}{k} = \frac{8}{7}$. Solid lines correspond to the Hashin–Shtrikman bounds (3)-(5); dashed and dotted lines corresponds to (87) and (86), respectively.

$$-\frac{\left(\frac{2\mu}{k}+1\right)^2(\mu-\mu_1)^3}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^3} + \frac{2\left(\frac{2\mu}{k}+1\right)(\mu-\mu_1)^3}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^3}$$

$$\left.-\frac{32.1697(\mu-\mu_1)^3}{\left(\mu_1\left(\frac{2\mu}{k}+1\right)+\mu\right)^3}\right) f^3. \qquad (87)$$

The dependencies of $\frac{\mu_e}{\mu}$ on the concentration are displayed in Fig. 4. The concentration is restricted by $f = 0.2$ in order to prevent overlapping of disks with fixed centers.

## 7. Conclusion

Analytical formulae (82) and (85) for the effective constants are deduced up to $O(f^4)$ for an arbitrary 2D macroscopically isotropic composite with circular inclusions. The principal step is the proper definition of the convergent series arisen in the second order terms $O(f^2)$. The calculated terms $O(f^2)$ do not depend on the location of inclusions whilst the third order terms do. In particular, formulae (82) and (85) imply that any SCM is valid only up to $O(f^3)$ for macroscopically isotropic composites. The following logic arguments were given in Section 5 of Mityushev and Rylko (2013). Any SCM for random composites is based on the consideration of the averaged structures and introduction of the unknown effective tensor $C_e$ without any precise description of the geometry. Usually, it is just said that a random composite is considered. However, a random composite is determined by the corresponding probabilistic geometric distribution, say $P$. Therefore, $C_e = C_e(P)$. In the case of disks, $C_e(P)$ depends on the distribution $P$ of the centers $a_k$ (the set $\{a_k\}_{k\in\mathcal{J}}$ can be treated as a random variable here). Any known final formula of the SCM is independent on $(a_k - a_m)$. Therefore, the SCM must yield universal formulae valid for any $P$. One can see from (82)–(85) that this universality ends on the term $f^2$ for macroscopically isotropic composites. The third order terms depend on the $e$-sums which in turn depend on $(a_k - a_m)$.

There are modifications of the SCM taking into account 2-, 3-, and $n$- particles interactions. In this case, a finite set of $(a_k - a_m)$ might be taken into account. However, this finite set does not produce high order formulae since actually it is equivalent to Maxwell's approach applied to dilute clusters containing $n$ inclusions. Hence, it gives only first order approximation for clusters (see Mityushev and Adler (2002)). Of course, combinations of the SCM with other methods can increase the precision when the corresponding manipulations can be made within the justified precision. It can be also seen that any SCM holds up to $O(f^2)$ for general anisotropic composites.

In the present paper, first, we determine the local elastic field for a finite number $n$ of inclusions on the plane by means of the functional Eqs. (12) and (13). The series method is outlined to analytically solve these equations with a prescribed precision in the powers of $r^2$. For any finite $n$, the obtained solution can give formulae for the effective constants only for the disk clusters within the first order concentration Mityushev and Adler (2002); Mityushev and Rylko (2013). In order to discuss higher concentrated random composites we investigate the limit $n \to \infty$ by the Eisenstein summation method of the conditionally convergent series.

Though, the final formulae (82) and (85) are written up to $O(f^4)$ the method does not have any restriction to get more precise formulae. We include an approximate analytical formula for the hexagonal array valid up to $O(f^9)$ (see Sec.3 of Supplement). The main computational difficulty for general composites is a convenient presentation of long analytical formulae. Another computational difficulty is related to the number of inclusions per cell, $N$, which has to be taken sufficiently large[5]. As an example, we calculate the effective constants for the composite displayed in Figs. 2 and 3. General investigation of random composites can be performed by the Monte Carlo method following Czapla et al. (2012) and Mityushev and Nawalaniec (2015). These questions will be discussed in a separate paper by advanced symbolic-numerical computations.

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## Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.ijsolstr.2016.06.034

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[5] It is not known how large at this moment

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