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Composites with invisible inclusions: Eigenvalues of $\mathbb{R}$-linear problem

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A new eigenvalue $\mathbb{R}$-linear problem arisen in the theory of metamaterials and neutral inclusions is reduced to integral equations. The problem is constructively investigated for circular non-overlapping inclusions. An asymptotic formula for eigenvalues is deduced when the radii of inclusions tend to zero. The nodal domains conjecture related to univalent eigenfunctions is posed. Demonstration of the conjecture allows to justify that a set of inclusions can be made neutral by surrounding it with an appropriate coating.

Key words: invisible inclusions, $\mathbb{R}$-linear problem, metamaterials, univalent eigenfunction.

1 Introduction

Local fields in fibrous composites are described by solutions of the Riemann–Hilbert and the $\mathbb{R}$-linear problems for multiply connected domains [16, 24, 25, 29–31]. The physical properties of the components of traditional composites are expressed in terms of the positive constants, c.f., conductivity, permeability, permittivity etc.

Recently, materials having negative physical constants were discovered. It concerns dielectric-magnetic materials displaying a negative index of refraction [1, 4, 10, 26, 27]. Mathematical modelling of metamaterials and neutral (invisible) inclusions were discussed in [2, 3, 13–15, 17, 20] and works cited therein. It was proved in [14, 15] that inclusions can be made neutral to all the directions of uniform fields if they are ellipses or ellipsoids. The paper [13] contains a general observation that any shaped inclusion with a smooth boundary can be made neutral to any fixed direction of the uniform field by surrounding it with an appropriate coating. This result is based on the study of the eigenvalues of the $\mathbb{R}$-linear problem for a doubly connected domain D when the spectral parameter is assigned only to one component of ∂D. Such a problem can be considered as a modification of the result [22, 32] devoted to eigenvalues of the $\mathbb{R}$-linear problem with the same spectral parameter in each component of the boundary.

The discussed eigenvalue problem differs from the classic problem when the spectral parameter λ enters into equation, for instance, $\Delta u + \lambda u = 0$ [9]. Our eigenvalue problem is similar to the Steklov problem [19] when $\Delta u = 0$ in D and $u = \lambda \frac{\partial u}{\partial n}$ on the boundary. Similar mixed boundary-spectral $\mathbb{R}$-linear problems were studied in [6] by reduction to integral equations and in [5] by variational methods.

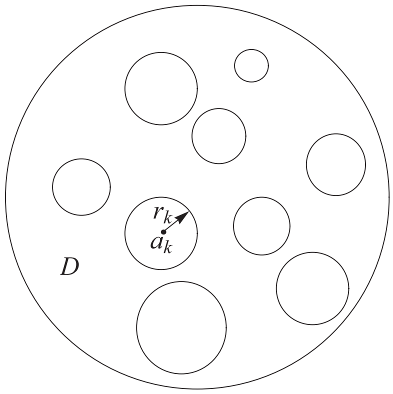

FIGURE 1. Multiply connected domain D with circular boundaries.

In the present paper, the general eigenvalue $\mathbb{R}$-linear problem arisen in the theory of metamaterials is reduced to integral equations. For circular non-overlapping inclusions, the considered problem is reduced to functional equations. An asymptotic formula for eigenvalues is deduced when the radii of inclusions tend to zero. The nodal domains conjecture related to univalent eigenfunctions is posed.

2 Eigenvalues $\mathbb{R}$-linear problem and integral equations

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. Consider n non-overlapping simply connected domains D_k $(k = 1, 2, \ldots, n)$ lying in the unit disk U and the multiply connected domain $D = U \setminus \cup_{k=1}^{n} (D_k \cup \partial D_k)$ (see Figure 1 with circular inclusions). Let D_0 denote the exterior of the closed unit disk to the extended complex plane. Let the boundary of each D_k $(k = 0, 1, \ldots, n)$ be a counter clock-wise oriented smooth simple curve Γ_k including the unit circle Γ_0.

Given Hölder continuous functions $a_k(t)$, $b_k(t)$ on Γ_k satisfying the inequality $|a_k(t)| > |b_k(t)|$ $(k = 0, 1, 2, \ldots, n)$. It is assumed that the winding number (index) of each $a_k(t)$ vanishes [12]. To find functions $\varphi_k(z)$ analytic in D_k, respectively, continuous in the closures of the considered domains and to find a complex constant $\lambda \neq 0$ such that the following $\mathbb{R}$-linear conditions are fulfilled

$$\varphi(t) = a_k(t)\varphi_k(t) + b_k(t)\overline{\varphi_k(t)}, \quad t \in \Gamma_k, \quad k = 1, 2, \ldots, n, \tag{2.1}$$

$$\varphi(t) = \overline{\lambda}\, a_0(t)\varphi_0(t) + b_0(t)\overline{\varphi_0(t)}, \quad |t| = 1. \tag{2.2}$$

It is assumed that the unknown function $\varphi_0(z)$ is analytic in $|z| > 1$ continuous in $|z| \geqslant 1$ and vanishes at infinity:

$$\varphi_0(\infty) = 0. \tag{2.3}$$

A non-zero function $\varphi_0(z)$ satisfying (2.1)–(2.3) is called the eigenfunction and the corresponding constant λ the eigenvalue of the problem. The function $\varphi_0(z)$ is distinguished from others, since the function $\omega(z) = \overline{\varphi_0\left(\frac{1}{\overline{z}}\right)}$, $|z| \geqslant 1$, plays the key role in applications.

We consider the problem (2.1)–(2.3) with the constant coefficients $a_k(t) = 1$, $b_k(t) = -\rho_k$, where $|\rho_k| < 1$ $(k = 1, 2, \ldots, n)$ and $a_0(t) = 1$, $b_0(t) = -1$. Then, (2.1)–(2.3) become

$$\varphi(t) = \varphi_k(t) - \rho_k \overline{\varphi_k(t)}, \quad t \in \Gamma_k, \quad k = 1, 2, \ldots, n, \tag{2.4}$$

$$\varphi(t) = \overline{\lambda} \varphi_0(t) - \overline{\varphi_0(t)}, \quad |t| = 1, \tag{2.5}$$

$$\varphi_0(\infty) = 0. \tag{2.6}$$

The problem (2.4)–(2.6) can be stated in terms of harmonic functions [23, 28]. The condition (2.5) for real λ up to an additive constant can be written in the form

$$u = (\lambda - 1) u_0, \quad \frac{\partial u}{\partial n} = (\lambda + 1) \frac{\partial u_0}{\partial n}, \quad |t| = 1, \tag{2.7}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative to the unit circle, $u = \mathrm{Re}\ \varphi$ and $u_0 = \mathrm{Re}\ \varphi_0$. Two real equations (2.7) are equivalent to the complex one (2.5) up to an additive arbitrary constant [23]. The real part of (2.5) yields the first equation (2.7). The imaginary part of (2.5) after the tangent differentiation and application of the Cauchy–Riemann equations gives the second equation (2.7). Along similar lines the condition (2.4) for real ρ_k becomes

$$u = (1 - \rho_k) u_0, \quad \frac{\partial u}{\partial n} = (1 + \rho_k) \frac{\partial u_0}{\partial n}, \quad t \in \Gamma_k. \tag{2.8}$$

The problem (2.7), (2.8), hence the problem (2.4)–(2.6), can be considered as the ideal contact conductivity problem for the 2D composite shown in Figure 1 with the conductivities $\frac{1+\rho_k}{1-\rho_k}$ in D_k and $\frac{\lambda+1}{\lambda-1}$ in $|z| > 1$, respectively. External sources are not applied to medium. It follows from Bojarski's theorem [7,8] that the eigenvalues of the problem (2.4)–(2.6) satisfy the inequality $|\lambda| \leqslant 1$. The case $\lambda = 1$ $(\lambda = -1)$ corresponds to the perfect conductor (isolator) in $|z| > 1$ and yields only constant potentials. The case $-1 < \lambda < 1$ corresponds to a metamaterial in $|z| > 1$ when potentials can be non-constant.

From other side, the same $\mathbb{R}$-linear problem (2.4)–(2.6) describes the flow around neutral inclusions with the standard contrast parameters ρ_k $(k = 1, 2, \ldots, n)$ in the conformally transformed plane. Here, the external parallel field is applied to medium. More precisely, let the univalent function $w = \omega(z) = \overline{\varphi_0\left(\frac{1}{\overline{z}}\right)}$ map the unit disk onto a domain G. This conformal mapping determines the shapes of the inclusions $\omega(\Gamma_k)$ $(k = 1, 2, \ldots, n)$ and of the corresponding neutral coating $\omega(\Gamma_0)$. An eigenvalue λ determines the positive coating conductivity depending on the direction of the external uniform field (for details see [13]).

Following [13] we reduce the $\mathbb{R}$-linear problem (2.4)—(2.6) to integral equations. First, introduce the Cauchy operator in the space of the Hölder continuous functions on the boundary $\partial D = \cup_{k=1}^{n} (-\Gamma_k) \cup \Gamma_0$ [12,23]

$$(S\mu)(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\mu(t)\, dt}{t - z}, \quad z \in \cup_{k=0}^{n} D_k. \tag{2.9}$$

It follows from the properties of the Cauchy-type integral [12] that $(S\varphi)(z) = 0$ since $\varphi(z)$ is analytic in D and $z \notin D$. Application of the operator S to the conditions (2.4), (2.5) yields

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{[\overline{\lambda}\varphi_0(t)-\overline{\varphi_0(t)}]dt}{t-z}=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\Gamma_k}\frac{[\varphi_k(t)-\rho_k\overline{\varphi_k(t)}]dt}{t-z},\quad z\in\cup_{k=0}^{n}D_k. \tag{2.10}$$

Using formulae [12]

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{\varphi_k(t)\,dt}{t-z}=\varphi_k(z),\quad z\in D_k\quad\text{and}\quad\frac{1}{2\pi i}\int_{\Gamma_k}\frac{\varphi_k(t)\,dt}{t-z}=0,\quad z\in D_m\ (m\neq k) \tag{2.11}$$

we arrive at the following system of integral equations with the spectral parameter λ

$$\overline{\lambda}\varphi_0(z)-\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\overline{\varphi_0(t)}dt}{t-z}+\sum_{k=1}^{n}\frac{\rho_k}{2\pi i}\int_{\Gamma_k}\frac{\overline{\varphi_k(t)}dt}{t-z}=0,\quad z\in D_0, \tag{2.12}$$

$$\varphi_m(z)-\sum_{k=1}^{n}\frac{\rho_k}{2\pi i}\int_{\Gamma_k}\frac{\overline{\varphi_k(t)}dt}{t-z}+\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\overline{\varphi_0(t)}dt}{t-z}=0,\quad z\in D_m\ (m=1,2,\dots,n).$$

The integral equations (2.12) can be written in the contour ∂D by pass to the limit $z\to\tau\in\partial D$ and application of the Sochocki (Sokhotskij–Plemelj) formulae [12]. The obtained contour integral equations can be considered in the Banach spaces of the Hölder continuous functions and in L_p where the singular integrals are bounded operators [12,23]. Equations (2.12) can be also considered in the space $\mathscr{C}^1(\cup_{m=1}^{n}D_m)$ endowed with the norm (3.4) and in $\mathscr{H}^2(\cup_{m=1}^{n}D_m)$ with the norm (3.6) discussed in the next section. It is worth noting that the integral equations (2.12) differ from the integral equations considered in [3].

3 Functional equations

In the present and next sections, we consider the $\mathbb{R}$-linear problem (2.4)–(2.6) when Γ_k are circles $|t-a_k|=r_k$. Following [23,24] we reduce the problem to a system of functional equations.

Let

$$z_{(m)}^{*}=\frac{r_m^2}{\overline{z-a_m}}+a_m,$$

denote the inversion with respect to the circle $|z-a_k|=r_k$. Introduce the function

$$\Phi(z):=\begin{cases}\varphi_k(z)+\sum_{m\neq k}\rho_m\overline{\varphi_m\left(z_{(m)}^{*}\right)}+\overline{\varphi_0\left(\frac{1}{\overline{z}}\right)}, & \\ \qquad\qquad\qquad\qquad |z-a_k|\leqslant r_k,\quad k=1,2,\dots,n, & \\ \varphi(z)+\sum_{m=1}^{n}\rho_m\overline{\varphi_m\left(z_{(m)}^{*}\right)}+\overline{\varphi_0\left(\frac{1}{\overline{z}}\right)},\quad z\in D, & \\ \overline{\lambda}\varphi_0(z)+\sum_{m=1}^{n}\rho_m\overline{\varphi_m\left(z_{(m)}^{*}\right)},\quad |z|\geqslant 1. & \end{cases}$$

analytic in D_k $(k=0,1,\dots,n)$ and D. Calculate the jump of $\Phi(z)$ across the circle $|t-a_k| = r_k$

$$\Delta_k := \Phi^+(t) - \Phi^-(t),\ |t-a_k| = r_k,$$

where $\Phi^+(t) := \lim_{z\to t\ z\in D} \Phi(z)$, $\Phi^-(t) := \lim_{z\to t\ z\in D_k} \Phi(z)$. Application of (2.4) gives $\Delta_k = 0$. Similar arguments for the jump Δ_0 of $\Phi(z)$ across the unit circle yield $\Delta_0 = 0$. It follows from the principle of analytic continuation that $\Phi(z)$ is analytic in the extended complex plane. Then, Liouville's theorem implies that $\Phi(z)$ is a constant. Calculation of this constant as $\Phi(\infty)$ and using of (2.6) yields

$$\Phi(z) = \sum_{m=1}^{n} \rho_m \overline{\varphi_m(a_m)}. \tag{3.1}$$

The definition of $\Phi(z)$ in $|z-a_k| \leqslant r_k$ and $|z| \geqslant 1$ leads to the following system of functional equations

$$\begin{aligned} \varphi_k(z) = -\textstyle\sum_{m\neq k} \rho_m \left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m(a_m)}\right] + \rho_k\overline{\varphi_k(a_k)} - \overline{\varphi_0\left(\tfrac{1}{\overline{z}}\right)}&, \\ |z-a_k| \leqslant r_k,\quad k = 1,2,\ldots,n&, \\ \overline{\lambda}\varphi_0(z) = -\textstyle\sum_{m=1}^{n} \rho_m \left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m(a_m)}\right],\ |z| \geqslant 1.& \end{aligned} \tag{3.2}$$

Exclusion of $\varphi_0(z)$ from (3.2) yields the system

$$\begin{aligned} \varphi_k(z) = -\textstyle\sum_{m\neq k} \rho_m \left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m(a_m)}\right] + \rho_k\overline{\varphi_k(a_k)}& \\ +\tfrac{1}{\lambda}\textstyle\sum_{m=1}^{n} \overline{\rho_m}\left[\varphi_m\left(a_m + \frac{r_m^2 z}{1-\overline{a_m}z}\right) - \varphi_m(a_m)\right]&, \\ |z-a_k| \leqslant r_k,\quad k = 1,2,\ldots,n.& \end{aligned} \tag{3.3}$$

We will assume that $\varphi_k(z)$ are analytic in $|z-a_k| < r_k$ and continuously differentiable in $|z-a_k| \leqslant r_k$ due to the physical treatment of $\varphi_k(z)$ as complex potentials. Introduce the space of functions $\mathscr{C}^1(\cup_{m=1}^n D_m)$ analytic in the non-connected domain $\cup_{m=1}^n D_m$ and continuously differentiable in its closure with the norm

$$\|\phi\|_{\mathscr{C}^1} = \max_{m=1,2,\ldots,n}\ \max_{|z-a_m|=r_m} |\varphi_m(z)| + \max_{m=1,2,\ldots,n}\ \max_{|z-a_m|=r_m} |\varphi_m'(z)|, \tag{3.4}$$

where $\phi(z) = \varphi_m(z)$ in $|z-a_m| \leqslant r_m$. One can write the system (3.3) as an equation in the Banach space $\mathscr{C}^1(\cup_{m=1}^n D_m)$

$$\phi = A\phi + \frac{1}{\lambda}B\phi, \tag{3.5}$$

where the operators A and B are introduced in accordance with (3.3) for shortness. Equation (3.5) can be considered in the Hilbert space $\mathscr{H}^2(\cup_{m=1}^n D_m)$ of functions $\phi(z) = \varphi_k(z)$ which belong to the Hardy space in the disks $|z-a_k| < r_k$ with the norm [11]

$$\|\phi\|_{H^2} = \left(\sum_{m=1}^{n} \sup_{R<r_m} \frac{1}{2\pi}\int_0^{2\pi} |\varphi_m(a_m + Re^{i\theta})|^2 d\theta\right)^{\frac{1}{2}}. \tag{3.6}$$

It follows from [23] that the operators A and B are compact in the considered spaces and the operator $I-A$ is invertible where I denotes the identity operator. Then, equation (3.5) is equivalent to the eigenvalue problem

$$\lambda\phi = (I-A)^{-1}B\phi, \tag{3.7}$$

where the operator $(I-A)^{-1}B$ is compact in the space $\mathscr{H}^2(\cup_{m=1}^n D_m)$. Therefore, the eigenvalue problem (3.3) can be written in the form of the eigenvalue problem (3.7) for a compact operator in the Hilbert space. Let $\phi \in \mathscr{H}^2(\cup_{m=1}^n D_m)$ be its eigenfunction. Then, Pumping principle [23] implies that ϕ actually belongs to $\mathscr{C}^1(\cup_{m=1}^n D_m)$ (even to $\mathscr{C}^\infty$). It is based on the following arguments. For a fixed k, every function $\overline{\varphi_m\left(z_{(m)}^*\right)}$ $(m \neq k)$ is analytic in $|z-a_m| > r_m$ and $\overline{\varphi_0\left(\frac{1}{\overline{z}}\right)}$ in $|z| < 1$. The union of these domains contains the closed disk $|z-a_k| \leqslant r_k$. Hence, the right part of (3.3) is analytic in $|z-a_k| \leqslant r_k$. Therefore, the left part containing the function $\varphi_k(z)$, is also analytic in $|z-a_k| \leqslant r_k$.

Instead of the functional equations (3.3) we consider equations in the space $\mathscr{C}(\cup_{m=1}^n D_m)$ associated with continuous functions obtained by differentiation of (3.3)

$$\begin{aligned}\psi_k(z) = {} & \sum\nolimits_{m\neq k} \rho_m \tfrac{r_m^2}{(z-a_m)^2}\, \overline{\psi_m\left(z_{(m)}^*\right)} \\ & + \tfrac{1}{\lambda} \sum\nolimits_{m=1}^n \overline{\rho_m} \tfrac{r_m^2}{(1-\overline{a_m}z)^2}\, \psi_m\left(a_m + \tfrac{r_m^2 z}{1-\overline{a_m}z}\right), \\ & \qquad |z-a_k| \leqslant r_k, \quad k = 1,2,\ldots,n,\end{aligned} \tag{3.8}$$

where $\psi_k(z) = \varphi_k'(z)$. One can see from the second equation (3.2) that $\varphi_0(z)$ does not depend on $\varphi_m(a_m)$. Therefore, one can first solve the system (3.8) and determine

$$\varphi_0'(z) = \frac{1}{\overline{\lambda}} \sum_{m=1}^n \frac{\rho_m r_m^2}{(z-a_m)^2}\, \overline{\psi_m\left(z_{(m)}^*\right)}, \quad |z| \geqslant 1. \tag{3.9}$$

The function $\varphi_0(z)$ is uniquely found from (3.9) by integration

$$\varphi_0(z) = -\int_z^\infty \varphi_0'(\zeta)d\zeta, \quad |z| \geqslant 1. \tag{3.10}$$

Remark 1 The eigenvalues of the Laplace operator form an increasing sequence [9]. In our case, the eigenvalue problem (3.7) or (3.8) is addressed to a compact operator. Therefore, the absolute values of eigenvalues decrease to zero [18].

4 Asymptotic solution of functional equations

In the present section, we find asymptotic solutions of the systems (3.2) and (3.3) when $r = \max_{k=1,2,\ldots,n} r_k$ tends to zero. The parameters $\nu_k = \frac{r_k^2}{r^2}$ are considered as values for which $0 < \nu_k \leqslant 1$ including the limit case, as $r \to 0$.

Lemma 1 *The eigenvalues $\lambda = \lambda(r)$ satisfy the asymptotic relation*

$$\lambda(r) = r^2 \lambda_0(r), \quad as\ r \to 0, \tag{4.1}$$

where the function $\lambda_0(r)$ is bounded as r tends to zero.

Proof The functions $\psi_k(z)$ analytic in $|z - a_k| < r_k$ are represented by their Taylor series

$$\psi_k(z) = \sum_{l=0}^{\infty} \alpha_l^{(k)} \left(\frac{z - a_k}{r} \right)^l, \ |z - a_k| < r_k, \quad k = 1, 2, \ldots, n. \tag{4.2}$$

Here, the coefficients $\alpha_l^{(k)}$ are normalized in such a way that they are bounded as $r \to 0$. For definiteness, the eigenfunctions are supposed to be normalized as

$$\|\phi\|^2_{\mathscr{H}^2} = \sum_{m=1}^{n} \sum_{l=0}^{\infty} v^l |\alpha_l^{(m)}|^2 = 1, \tag{4.3}$$

where $\phi(z) = \psi_m(z)$ in $|z - a_m| \leqslant r_m$.

Using (4.2) we write equation (3.8) up to $O\left(\frac{r^4}{\lambda(r)}\right)$ considering $\lambda(r)$ in general form since its asymptotic behavior has been not known yet

$$\begin{aligned} \psi_k(z) = r^2 \textstyle\sum_{m \neq k} \frac{\rho_m v_m}{(z - a_m)^2} \left[\overline{\alpha_0^{(m)} + r \frac{v_m \alpha_1^{(m)}}{z - a_m}} \right] \\ + \frac{r^2}{\lambda(r)} \textstyle\sum_{m=1}^{n} \frac{\overline{\rho_m} v_m}{(1 - \overline{a_m} z)^2} \left[\alpha_0^{(m)} + r \frac{v_m \alpha_1^{(m)} z}{1 - \overline{a_m} z} \right] + O\left(\frac{r^4}{\lambda(r)}\right), \\ |z - a_k| \leqslant r_k, \quad k = 1, 2, \ldots, n. \end{aligned} \tag{4.4}$$

Substitute $z = a_k$ into (4.4) and reduce the order of approximation to $O\left(\frac{r^2}{\lambda(r)}\right)$

$$\begin{aligned} \alpha_0^{(k)} = r^2 \textstyle\sum_{m \neq k} \frac{\rho_m v_m}{(a_k - a_m)^2} \overline{\alpha_0^{(m)}} + \frac{r^2}{\lambda(r)} \textstyle\sum_{m=1}^{n} \frac{\overline{\rho_m} v_m}{(1 - \overline{a_m} a_k)^2} \alpha_0^{(m)} + O\left(\frac{r^3}{\lambda(r)}\right), \\ k = 1, 2, \ldots, n. \end{aligned} \tag{4.5}$$

Differentiate equations (4.4) and substitute $z = a_k$ into the result multiplied by r

$$\begin{aligned} \alpha_1^{(k)} = -2r^3 \textstyle\sum_{m \neq k} \frac{\rho_m v_m}{(a_k - a_m)^3} \overline{\alpha_0^{(m)}} + 2 \frac{r^3}{\lambda(r)} \textstyle\sum_{m=1}^{n} \frac{\overline{\rho_m} v_m \overline{a_m}}{(1 - \overline{a_m} a_k)^3} \alpha_0^{(m)} + O\left(\frac{r^4}{\lambda(r)}\right), \\ k = 1, 2, \ldots, n. \end{aligned} \tag{4.6}$$

This procedure can be continued to get the next equations for $\alpha_l^{(k)}$ $(l = 3, 4, \ldots)$.

We now prove that $\frac{r^2}{\lambda(r)}$ cannot tend to zero as $r \to 0$. If it is not so, then (4.5) implies that $\alpha_0^{(k)}$ tends to zero as $r \to 0$. Then, equation (4.6) implies that $\alpha_1^{(k)}$ tends to zero as $r \to 0$ and so forth $\alpha_l^{(k)} \to 0$ for all l. This contradicts to the normalization (4.3).

The lemma is proved. □

It follows form Lemma 1 that the maximally possible absolute value of an eigenvalue for sufficiently small r can be found in the form $\lambda = r^2 \mu + o(r^2)$, where μ is a non-zero constant. Take the main terms of (4.5) and write equations up to $O(r)$

$$\mu\alpha_0^{(k,0)} = \sum_{m=1}^{n} \frac{\overline{\rho_m}\nu_m}{(1-\overline{a_m}a_k)^2}\alpha_0^{(m,0)}, \quad k = 1,2,\ldots,n, \tag{4.7}$$

where $\alpha_0^{(k,0)} = \alpha_0^{(k)} + O(r)$. Introduce the matrix F whose elements have the form

$$F_{mk} = \frac{\overline{\rho_m}\nu_m}{(1-\overline{a_m}a_k)^2}. \tag{4.8}$$

The eigenvalues μ of the linear algebraic system (4.7) are solution of the polynomial equation

$$\det(\mu I - F) = 0, \tag{4.9}$$

where I stands for the identity matrix.

If $\rho_m = \rho \in \mathbb{R}$ for any m, the matrix (4.8) is self-adjoint. In this case, equation (4.9) has exactly n real roots counted with multiplicity.

The eigenfunctions can be constructed up to $O(r)$ by (4.4). Let μ be a simple eigenvalue and $\mathbf{v} = (\alpha_0^{(1,0)}, \alpha_0^{(2,0)}, \ldots, \alpha_0^{(n,0)})$ be the corresponding eigenvector of the linear algebraic system (4.7). Then, (3.9), (3.10) yield

$$\varphi_0(z) = -r^2 \sum_{m=1}^{n} \frac{\rho_m \nu_m}{z - a_m}\, \overline{\alpha_0^{(m,0)}}, \quad |z| \geqslant 1. \tag{4.10}$$

Example 1 ([21]) Let $n = 1$ and $\Gamma_1 = \{t \in \mathbb{C} : |t| = r\}$ with $0 < r < 1$ in the problem (2.4)–(2.6). All solutions of this problem have the following form

$$\varphi_1^{(p)}(z) = z^p, \quad \varphi_0^{(p)}(z) = -\frac{1}{z^p}, \quad \varphi^{(p)}(z) = z^p - \frac{\rho r^{2p}}{z^p}, \quad \lambda_p = \overline{\rho} r^{2p}, \quad p = 1,2,\ldots, \tag{4.11}$$

where the normalization $\|\varphi_1^{(k)}\|^2_{\mathscr{H}^2} = 1$ is chosen in accordance with (4.3).

The case $p = 1$ in Example 1 corresponds to (4.10) with $n = 1$, $a_1 = 0$ and $\alpha_0^{(m,0)} = 1$. The function, important in applications to metamaterials, $\omega_1(z) = \overline{\varphi_0^{(1)}\left(\frac{1}{\overline{z}}\right)} = z$ is univalent in the unit disk and determines a circle neutral inclusion with an annulus coating with a conductivity determined by $\lambda_1 = \overline{\rho} r^2$ [13].

Example 2 Let $n = 2$, $a_1 = a$, $a_2 = -a$ where a be a positive number; $\Gamma_1 = \{t \in \mathbb{C} : |t - a| = r\}$ and $\Gamma_2 = \{t \in \mathbb{C} : |t + a| = r\}$ where $a + r < 1$; $\rho_1 = \rho_2 \equiv \rho$. In this case, the system (4.7) becomes

$$\begin{aligned} \mu\alpha_0^{(1,0)} &= \overline{\rho}\left[\tfrac{1}{(1-a^2)^2}\alpha_0^{(1,0)} + \tfrac{1}{(1+a^2)^2}\alpha_0^{(2,0)}\right], \\ \mu\alpha_0^{(2,0)} &= \overline{\rho}\left[\tfrac{1}{(1+a^2)^2}\alpha_0^{(1,0)} + \tfrac{1}{(1-a^2)^2}\alpha_0^{(2,0)}\right]. \end{aligned} \tag{4.12}$$

The eigenvalues and eigenvectors of (4.12) have the form

$$\mu_1 = \overline{\rho}\frac{2(1+a^4)}{(1-a^4)^2},\ \mathbf{v}_1 = (1,1); \quad \mu_2 = \overline{\rho}\frac{4a^2}{(1-a^4)^2},\ \mathbf{v}_2 = (-1,1). \tag{4.13}$$

The corresponding functions $\omega_p(z) = \overline{\varphi_0^{(p)}\left(\frac{1}{\overline{z}}\right)}$ are given by the approximate analytical formulae up to a multiplier

$$\omega_1(z) = -\rho r^2 \frac{2z}{1-a^2z^2}, \quad \omega_2(z) = \rho r^2 \frac{2az^2}{1-a^2z^2}.$$

One can see that the function $\omega_1(z)$ is univalent in the unit disks. It corresponds to the maximal $|\lambda_1| = r^2|\mu_1|$.

5 Discussion

The above study and examples enables us to make the following:

Conjecture Let ρ_k be given real numbers. Then, all eigenvalues of the problem (2.4)–(2.6) are real. The set of eigenvalues is countable or finite. Let $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots$. Then, the corresponding eigenfunctions $\omega_p(z) = \overline{\varphi_0^{(p)}\left(\frac{1}{\overline{z}}\right)}$ $(p = 1, 2, \ldots)$ satisfy inequality

$$wind_{|z|=1}\omega_p(z) \leqslant p, \tag{5.1}$$

where the winding number (or index [12]) is defined as

$$wind_{|z|=1}f(z) = \frac{1}{2\pi i}\int_{|z|=1}\frac{f'(z)}{f(z)}\,dz.$$

One can see in Example 1 that

$$wind_{|z|=1}\omega_p(z) = wind_{|z|=1}\varphi_0^{(p)}(z) = p.$$

Moreover, $\max_k |\lambda_k| = |\lambda_1|$ and only the corresponding eigenfunction $\varphi_0^{(1)}(z)$ is conformal in $|z| > 1$.

Demonstration of Conjecture for $p = 1$ allows to justify that any shaped inclusion with a smooth boundary can be made neutral by surrounding it with an appropriate coating [13].

Conjecture recalls Courant's theorem [9] outlined below. Consider for definiteness the Dirichlet problem $u = 0$ on $\partial\Omega$ for equation $\Delta u = -\lambda u$ valid in a domain Ω. The set of eigenvalues consists of a sequence $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ (see Remark on page 6) and the corresponding eigenfunctions $u_1, u_2, \ldots$ constitute a complete orthonormal basis of $L_2(\Omega)$. The nodal set of a fixed u_p is defined as the set $\{z \in \Omega : \ u_p(z) = 0\}$. According to Courant's theorem [9] the number of nodal domains of u_p is less than or equal to p, for every $p = 1, 2, \ldots$.

Conjecture can be stated in terms of nodal domains of the eigenfunctions $Re\ \varphi_0^{(p)}(z)$ in $|z| > 1$ of the problem (2.4)–(2.6). Instead of (5.1) one can demand that the number of nodal domains of Re $\varphi_0^{(p)}(z)$ is less than or equal to $2p$, for every $p = 1, 2, \ldots$.

Let $\theta \in [0, 2\pi)$ denote the argument of the complex number z. It is easily seen that the nodal domains of the eigenfunctions $Re\ z^{-p} = |z|^{-p} \cos p\theta$ from Example 1 are $2p$ sectors separated by the rays $\arg z = \frac{\pi m}{p}$ where $m = 0, 1, \ldots, 2p - 1$.

The general problem (2.1)–(2.3) and its partial case (2.4)–(2.6) for general curves Γ_k have been not studied yet. Even in the case of n sufficiently small circular inclusions Conjecture has been not proven. It is reduced to the following seemingly simple question. Let points a_k $(k = 1, 2, \ldots, n)$ lie in the open unit disk and $\mathbf{v} = (\alpha_0^{(1,0)}, \alpha_0^{(2,0)}, \ldots, \alpha_0^{(n,0)})$ be eigenvectors of the eigenvalue problem (4.7). For which $\mathbf{v}$ is the function

$$\varphi_0(z) = \sum_{m=1}^{n} \frac{\rho_m r_m^2}{z - a_m} \overline{\alpha_0^{(m,0)}}, \tag{5.2}$$

univalent in $|z| > 1$ or $\omega(z) = \overline{\varphi_0\left(\frac{1}{\bar{z}}\right)}$ in $|z| < 1$? Does this $\mathbf{v}$ correspond to the maximal $|\lambda|$? This answer is interesting even for equal ρ_m and r_m when the number of eigenvectors holds n. It solves the problem of clouds of neutral inclusions.

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References

[1] Alu, A. & Engheta, N. (2005) Achieving transparency with plasmonic and metamaterial coatings. *Phys. Rev. E* **72**, 016623.

[2] Ammari H., Kang H., Lee H. & Lim M. (2013) Enhancement of near cloaking using generalized polarization tensors vanishing structures. Part I: The conductivity problem. *Comm. Math. Phys.* **317**, 253–266.

[3] Ammari H., Ciraolo G., Kang H., Lee H. & Milton G. W. (2013) Spectral theory of a Neumann-Poincaree-type operator and analysis of cloaking due to anomalous localized resonance. *Arch. Ration. Mech. Anal.* **208**, 667–692

[4] Asatryan A. A., Botten L. C., Fang K., Fan S. & McPhedran R. C. (2014) Two-dimensional Green's tensor for gyrotropic clusters composed of circular cylinders. *J. Opt. Soc. Am. A Opt. Image Sci. Vis.* **31**, 2294–303.

[5] Banuelos, R., Kulczycki, T., Polterovich, I. & Siudeja, B. (2010) Eigenvalue inequalities for mixed Steklov problems. In "Operator Theory and Its Applications: In Memory of V. B. Lidskii (1924–2008)" edited by M. Levitin, D. Vassiliev, AMS Providence, Rhode Island, pp. 19–34.

[6] Bogatyrev, A. (2000) Poincaré-Steklov integral equations and the Riemann monodromy problem. *Funct. Anal. Appl.* **34**, 86–97.

[7] Bojarski, B. (1960) On generalized Hilbert boundary-value problem. *Soobsch. Akad. Nauk Gruz. SSR* **25**, 385–390.

[8] Bojarski, B. & Mityushev, V. (2013) R-linear problem for multiply connected domains and alternating method of Schwarz. *J. Math. Sci.* **189**, 68–77.

[9] Chavel, I. (1984) *Eigenvalues in Riemannian Geometry*, London, Academic Press, pp. 7–24.

[10] Colquitt, D. J., Jones, I. S., Movchan, N. V., Movchan, A. B., Brun, M. & McPhedran, R. C. (2013) Making waves round a structured cloak: Lattices, negative refraction and fringes. *Proc. Roy. Soc.* **A469**, 20130218.

[11] Drygas, P. (2007) A functional-differential equation in a class of analytic functions and its application. *Aequationes Math.* **73**, 222–232.

[12] Gakhov, F. D. (1977) *Boundary Problems*, Moscow, Nauka, pp. 37–43.

[13] Jarczyk, P. & Mityushev, V. (2011) Neutral coated inclusions of finite conductivity. *Proc. R. Soc. Lond.* **A468**, 954–970.

[14] Kang H. & Lee H. (2014) Coated inclusions of finite conductivity neutral to multiple fields in two dimensional conductivity or anti-plane elasticity. *Euro. J. Appl. Math.* **25**, 329–338.

[15] Kang H., Lee H. & Sakaguchi, S. (2014) An over-determined boundary value problem arising from neutrally coated inclusions in three dimensions. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, to appear. arXiv 1501.07465

[16] Kalamkarov, A. L., Andrianov, I. V. & Danishevskyy, V. V. (2009) Asymptotic homogenization of composite materials and structures, *Appl. Mech. Rev.* **62**, 030802-1–20.

[17] Kerker, M. (1975) Invisible bodies. *J. Opt. Soc. Am.* **65**, 376–379.

[18] Kolmogorov, A. N. & Fomin, S. V. (1999) *Elements of the Theory of Functions and Functional Analysis*, New York, Dover Publications, pp. 246–253.

[19] Lebedev, V. I. & Agoshkov, V. I. (1983) *Poincaré - Steklov Operators and their Applications in Analysis*, Moscow, Akad. Nauk SSSR, Vychisl. TCentr Publ, pp. 6–78.

[20] Milton, G. W. & Serkov, S. K. (2001) Neutral coated inclusions in conductivity and anti-plane elasticity. *Proc. R. Soc. Lond.* **A457**, 1973–1997.

[21] Mityushev, V. V. (1984) *Boundary Value Problems and Functional Equations with Shifts in Domains*, Byelorussian State University, Minsk, PhD Thesis.

[22] Mityushev, V. V. (1992) Eigenvalues of the $\mathbb{R}$-linear problems. *Izvestia vuzov. Math.* **11**, 35–38.

[23] Mityushev, V. V. & Rogosin, S. V. (2000) *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions. Theory and Applications*, Chapman & Hall / CRC, Boca Raton, London, New York, Washington DC, pp. 133–201.

[24] Mityushev, V. V. (2011) Riemann-Hilbert problems for multiply connected domains and circular slit maps. *Comp. Meth. Func. Theory* **11**, 557–90.

[25] Mityushev, V. & Rylko, N. (2012) Optimal distribution of the non-overlapping conducting disks. *Multiscale Model. Simul.* **12**, 180–90.

[26] O'Neill, J., Selsil, Ö., McPhedran, R. C., Movchan, A. B. & Movchan, N. V. (2015) Active cloaking of inclusions for flexural waves in thin elastic plates. *Q. J. Mech. Appl. Math.* published online June 25, doi:10.1093/qjmam/hbv007.

[27] Poulton, C. G., Movchan, A. B., Movchan, N. V. & McPhedran, R. C. (2012) Analytic theory of defects in periodically structured elastic plates. *Proc. R. Soc.* **A471**, 1196–16.

[28] Rylko, N. (2000) Transport properties of a rectangular array of highly conducting cylinders. *J. Eng. Math.* **38**, 1–12.

[29] Rylko, N. (2008) Structure of the scalar field around unidirectional circular cylinders. *Proc. R. Soc.* **A464**, 391–407.

[30] Rylko, N. (2013) Effective anti-plane properties of piezoelectric fibrous composites. *Acta Mech.* **224**, 2719–34.

[31] Rylko, N. (2015) Fractal local fields in random composites. *Comput. Math. Appl.* **69**, 247–54.

[32] Schiffer, M. (1959) Fredholm eigen values of multiply-connected domains. *J. d'Anal. Math.* **9**, 211–69.