Poincaré $\boldsymbol{\alpha}$–series for classical Schottky groups and its applications

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Abstract

The Poincaré $\boldsymbol{\alpha}$–series ($\boldsymbol{\alpha} \in \mathbb{R}^n$) for classical Schottky groups is introduced and used to solve Riemann–Hilbert problems for n–connected circular domains. The classical Poincaré θ_2–series can be obtained from the $\boldsymbol{\alpha}$–series by the substitution $\boldsymbol{\alpha} = \mathbf{0}$. The real Jacobi inversion problem and its generalisations are investigated via the Poincaré $\boldsymbol{\alpha}$–series. In particular, it is shown that the Riemann theta–function coincides with the Poincaré $\boldsymbol{\alpha}$–series. Relations to conformal mappings to slit domains and the Schottky–Klein prime function are established. A fast algorithm to compute Poincaré series for disks close to each other is outlined.

1 Introduction

The θ_2–series of Poincaré associated to the classical Schottky groups is used in the constructive theory of analytic functions in multiply connected domains. Such objects of multiply connected domains as the harmonic measures [23, 26, 27], the Abelian functions [1, 4, 7, 8], the canonical conformal mappings [9, 11, 16, 28], the Christoffel–Schwarz formula [13, 14, 15, 17, 31], the Bergman kernel [20] can be constructed by the Poincaré series. These objects can be also considered in the equivalent form on the Schottky double. The Poincaré series have applications to extremal polynomials [5], to the generalized alternating method of Schwarz [24, 22, 37] and to composites [29]. The above objects are ultimately constructed for all circular multiply connected domains [27, 28, 31, 20] via the uniformly convergent θ_2–series of Poincaré [25]. The method of construction is based on Riemann–Hilbert problems and functional equations (without integral terms) [23, 26, 27]. It is worth noting that investigations based on the absolute convergence have geometrical restrictions on the location of the circular holes [36].

A method of functional equations [23, 26] gives more general series than the classical θ_2–series of Poincaré. In the present paper, such series, called the Poincaré $\boldsymbol{\alpha}$–series (shortly, the $\boldsymbol{\alpha}$–series), are systematically discussed. Here, $\boldsymbol{\alpha}$ is a constant vector from $\mathbb{R}^n$. If $\boldsymbol{\alpha} = \mathbf{0}$, we arrive at the classical Poincaré series. We solve Riemann–Hilbert problems in terms of the $\boldsymbol{\alpha}$–series and apply the results to the generalized Jacobi inversion problem. In order to simplify the presentation we consider the special Riemann–Hilbert problem with the winding number equal to the connectivity of the domain. A fast algorithm to compute Poincaré series is presented.

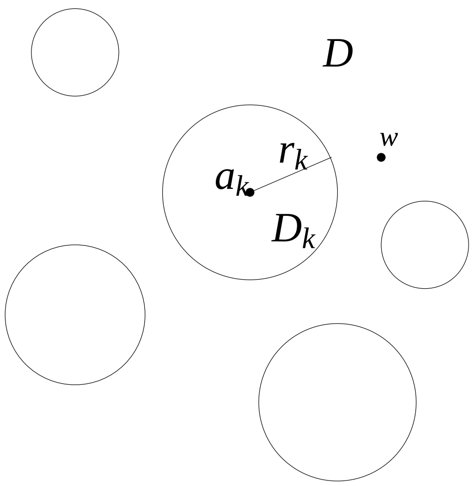

Figure 1: Multiply connected domain D with circular inclusions D_k.

2 Poincaré series for classical Schottky groups

Consider mutually disjointed disks $D_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$ in the complex plane $\mathbb{C}$ and the multiply connected domain D, the complement of the closed disks $|z - a_k| \leq r_k$ to the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (see Fig.1). Consider the inversion with respect to the circle $|z - a_k| = r_k$

$$z_{(k)}^* = \frac{r_k^2}{\overline{z - a_k}} + a_k.$$

Introduce the composition of successive inversions with respect to the circles

$$z^*_{(k_p k_{p-1}\ldots k_1)} := \left(z^*_{(k_{p-1}\ldots k_1)}\right)^*_{(k_p)}. \tag{2.1}$$

In the sequence $k_1, k_2, ..., k_p$ no two neighboring numbers are equal. The number p is called the *level* of the mapping. When p is even, these are Möbius transformations. If p is odd, we have anti-Möbius transformations, i.e., Möbius transformations in $\overline{z}$. Thus, these mappings can be written in the form

$$\begin{array}{rcl} \gamma_j(z) & = & (e_j z + b_j)\,/\,(c_j z + d_j)\,,\ p \in 2\mathbb{Z}, \\ \gamma_j(\overline{z}) & = & (e_j \overline{z} + b_j)\,/\,(c_j \overline{z} + d_j)\,,\ p \in 2\mathbb{Z}+1, \end{array} \tag{2.2}$$

where $e_j d_j - b_j c_j = 1$. Here $\gamma_0(z) := z$ (identical mapping with the level $p = 0$), $\gamma_1(\overline{z}) := z^*_{(1)}$,..., $\gamma_n(\overline{z}) := z^*_{(n)}$ (n simple inversions, $p = 1$), $\gamma_{n+1}(z) := z^*_{(12)}$, $\gamma_{n+2}(z) := z^*_{(13)}$,..., $\gamma_{n^2}(z) := z^*_{(n,n-1)}$ ($n^2 - n$ pairs of inversions, $p = 2$), $\gamma_{n^2+1}(\overline{z}) := z^*_{(121)}$, ... and so on. The set of the subscripts j of γ_j is ordered in such a way that the level p is increasing. The functions (2.2) generate a Schottky group $\mathcal{K}$. Thus, each element of $\mathcal{K}$ is presented in the form of the composition of inversions (2.1) or in the form of linearly ordered functions (2.2). All elements γ_j of the even levels generate a subgroup $\mathcal{E}$ of the group $\mathcal{K}$. The set of the elements γ_j of odd level $\mathcal{K}\backslash\mathcal{E}$ is denoted by $\mathcal{O}$.

Let $H(z)$ be a rational function. This following series is called the Poincaré θ_2–series

$$\theta_2(z) := \sum_{\gamma_j \in \mathcal{E}} H[\gamma_j(z)](c_j z + d_j)^{-2} \tag{2.3}$$

associated with the subgroup $\mathcal{E}$. It was proved in [25] that the series (2.3) converges uniformly in every compact subset not containing the limit points of $\mathcal{K}$ and poles of $H(\gamma_j(z))$. Moreover, it is an automorphic function:

$$\theta_2(z) = \theta_2[\gamma_j(z)](c_j z + d_j)^{-2}. \tag{2.4}$$

Using the inversions (2.1) instead of (2.2) we write (2.3) in the extended form. First, following [25] introduce the series

$$\Theta_2^{(1)}(z) = H(z) - \sum_{k=1}^{n} \overline{H[z^*_{(k)}]}(\overline{z^*_{(k)}})' + \sum_{k=1}^{n} \sum_{k_1 \neq k} H[z^*_{(k_1 k)}](z^*_{(k_1 k)})' - \tag{2.5}$$

$$\sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \overline{H[z^*_{(k_2 k_1 k)}]}(\overline{z^*_{(k_2 k_1 k)}})' + \ldots$$

and

$$\Theta_2^{(2)}(z) = H(z) + \sum_{k=1}^{n} \overline{H[z^*_{(k)}]}(\overline{z^*_{(k)}})' + \sum_{k=1}^{n} \sum_{k_1 \neq k} H[z^*_{(k_1 k)}](z^*_{(k_1 k)})' + \tag{2.6}$$

$$\sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \overline{H[z^*_{(k_2 k_1 k)}]}(\overline{z^*_{(k_2 k_1 k)}})' + \ldots$$

The Poincaré θ_2–series (2.3) can be written in the form

$$\theta_2(z) = \frac{1}{2}\left(\Theta_2^{(1)}(z) + \Theta_2^{(2)}(z)\right). \tag{2.7}$$

Let α_k $(k = 1, 2, \ldots, n)$ be real numbers from the segment $[0, 2\pi)$. Introduce the multi–index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and the series

$$\boldsymbol{\Theta}_2^{(1)}(z;\boldsymbol{\alpha}) = H(z) - \sum_{k=1}^{n} e^{2i\alpha_k} \overline{H[z^*_{(k)}]}(\overline{z^*_{(k)}})' + \sum_{k=1}^{n} \sum_{k_1 \neq k} e^{2i(\alpha_k - \alpha_{k_1})} H[z^*_{(k_1 k)}](z^*_{(k_1 k)})' - \tag{2.8}$$

$$\sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} e^{2i(\alpha_k - \alpha_{k_1} + \alpha_{k_2})} \overline{H[z^*_{(k_2 k_1 k)}]}(\overline{z^*_{(k_2 k_1 k)}})' + \ldots,$$

$$\boldsymbol{\Theta}_2^{(2)}(z;\boldsymbol{\alpha}) = H(z) + \sum_{k=1}^{n} e^{2i\alpha_k} \overline{H[z^*_{(k)}]}(\overline{z^*_{(k)}})' + \sum_{k=1}^{n} \sum_{k_1 \neq k} e^{2i(\alpha_k - \alpha_{k_1})} H[z^*_{(k_1 k)}](z^*_{(k_1 k)})' + \tag{2.9}$$

$$\sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} e^{2i(\alpha_k - \alpha_{k_1} + \alpha_{k_2})} \overline{H[z^*_{(k_2 k_1 k)}]}(\overline{z^*_{(k_2 k_1 k)}})' + \ldots$$

and

$$\boldsymbol{\theta}_2(z;\boldsymbol{\alpha}) = \frac{1}{2}\left[\boldsymbol{\Theta}_2^{(1)}(z;\boldsymbol{\alpha}) + \boldsymbol{\Theta}_2^{(2)}(z;\boldsymbol{\alpha})\right]. \tag{2.10}$$

We call the series (2.8)–(2.10) by the $\boldsymbol{\alpha}$–series. The series (2.8)–(2.10) uniformly converge in every compact subset not containing the limit points of $\mathcal{K}$ and poles of $H[\gamma_j(z)]$ [30]. If $\boldsymbol{\alpha} = (0, 0, \ldots, 0)$, we arrive at the classic Poincaré series (2.3).

Similar formulae take place for the Schottky–Klein prime function discussed in [1, 7, 8, 9, 10, 11]. Let ζ and w be fixed points of $(D \cup \partial D)\backslash\{\infty\}$. The following functions was introduced in [27, 28] (see formulae (40) and (41) in [28])

$$\omega_0(z, \zeta, w) = \ln \prod_{j=1}^{\infty} \mu_j(z, \zeta, w), \tag{2.11}$$

where

$$\mu_j(z,\zeta,w) = \begin{cases} \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)}, \text{ if } \gamma_j \in \mathcal{E}, \\ \\ \overline{\frac{\zeta-\gamma_j(\overline{w})}{\zeta-\gamma_j(\overline{z})}}, \text{ if } \gamma_j \in \mathcal{O}. \end{cases} \tag{2.12}$$

The multipliers $\mu_j(z,\zeta,w)$ in (2.11) are arranged in accordance with the increasing level of γ_j. The infinite product (2.11) converges uniformly in z in every compact subset of $(D \cup \partial D) \backslash (\{\infty\}, \{\zeta\}, \{w\})$. The justification of these assertions is based on the application of Lemma 3.1 from Sec.3 to the functional equations following [25, 27, 28]

$$\varphi_k(z) = -\sum_{m\neq k} \left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m\left(w^*_{(m)}\right)} \right] + \ln \frac{z-\zeta}{w-\zeta}, \ |z-a_k| \leq r_k, \ k = 1,\ldots,n. \tag{2.13}$$

Instead of (2.13) we can apply Lemma 3.1 to the following functional equations

$$\varphi_k(z) = \sum_{m\neq k} \left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m\left(w^*_{(m)}\right)} \right] + \ln \frac{z-\zeta}{w-\zeta}, \ |z-a_k| \leq r_k, \ k = 1,\ldots,n. \tag{2.14}$$

This justifies introduction of the function

$$\omega_1(z,\zeta,w) = \ln \prod_{j=1}^{\infty} \nu_j(z,\zeta,w), \tag{2.15}$$

where

$$\nu_j(z,\zeta,w) = \begin{cases} \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)}, \text{ if } \gamma_j \in \mathcal{E}, \\ \\ \overline{\frac{\zeta-\gamma_j(\overline{z})}{\zeta-\gamma_j(\overline{w})}}, \text{ if } \gamma_j \in \mathcal{O}. \end{cases} \tag{2.16}$$

Similar to (2.10) we introduce the function

$$\omega(z,\zeta,w) = \frac{1}{2}[\omega_0(z,\zeta,w) + \omega_1(z,\zeta,w)] = \frac{1}{2} \ln \prod_{\gamma_j \in \mathcal{E}\backslash\{\gamma_0\}} \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)}. \tag{2.17}$$

Hence, the following infinite product is correctly defined for z not equal to ζ, w and infinity

$$\Omega(z,\zeta,w) = \prod_{\gamma_j \in \mathcal{E}\backslash\{\gamma_0\}} \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)}. \tag{2.18}$$

Therefore, we can introduce the function of two variables

$$S(z,\zeta) = (\zeta - z)\Omega(\zeta, z, z)\Omega(z, \zeta, \zeta) = (\zeta - z) \prod_{\gamma_j \in \mathcal{E}\setminus\{\gamma_0\}} \frac{z - \gamma_j(\zeta)}{z - \gamma_j(z)} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(\zeta)}. \tag{2.19}$$

This is the famous Schottky–Klein function presented in the form of uniformly convergent product. More precisely, the uniform convergence is proved for $\Omega(\zeta, z, z)$ in the variable ζ in every compact subset of $(D \cup \partial D)\setminus(\{z\}, \{\infty\})$ and for $\Omega(z, \zeta, \zeta)$ in the variable z in every compact subset of $(D \cup \partial D)\setminus(\{\zeta\}, \{\infty\})$. The uniform convergence in the variable (z, ζ) in subsets of $\mathbb{C}^2$ could be proved by refined investigations of the corresponding functional equations.

Similar to (2.8)–(2.10) one can introduce $\boldsymbol{\alpha}$–prime functions

$$S(z,\zeta,\boldsymbol{\alpha}) = (\zeta - z) \prod_{\gamma_j \in \mathcal{E}\setminus\{\gamma_0\}} e^{2is_j(\boldsymbol{\alpha})} \frac{z - \gamma_j(\zeta)}{z - \gamma_j(z)} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(\zeta)}. \tag{2.20}$$

where for odd p

$$s_j(\boldsymbol{\alpha}) := \alpha_k - \alpha_{k_1} + \ldots + \alpha_{k_{p-1}} - \alpha_{k_p}. \tag{2.21}$$

The correspondence between j and $(k_p, k_{p-1}, \ldots k_1, k)$ in (2.21) is established via the numeration of the elements of $\mathcal{E}$, i.e., via the relation $\gamma_j(z) = z^*_{(k_p k_{p-1} \ldots k_1 k)}$.

3 Riemann–Hilbert problem

To find a function $\psi(z)$ analytic in D and continuously differentiable in $D \cup \partial D$ with the following Riemann–Hilbert boundary condition [27]

$$\mathrm{Im}\left[e^{-i\alpha_k} \frac{t - a_k}{r_k} \psi(t)\right] = 0, \quad |t - a_k| = r_k, \ k = 1, 2, \ldots, n. \tag{3.1}$$

It is assumed that the function $\psi(z)$ is normalized at infinity

$$\psi(\infty) = 1. \tag{3.2}$$

Let function $\varphi(z)$ be a primitive of $\psi(z)$, i.e., $\varphi'(z) = \psi(z)$. Then $\varphi(z)$ satisfies the Riemann–Hilbert boundary condition

$$\mathrm{Re}\,[e^{-i\alpha_k}\varphi(t)] = c_k, \quad |t - a_k| = r_k, \ k = 1, 2, \ldots, n, \tag{3.3}$$

where c_k are undetermined constants. In order to prove it, we consider the parametrisation of the circle $|t-a_k| = r_k$ with the natural arc parameter $s \in [0, 2\pi r_k)$

$$t(s) = a_k + r_k \exp\left(\frac{is}{r_k}\right). \tag{3.4}$$

One can see that the derivative can be written in the form

$$t'(s) = i\,\frac{t-a_k}{r_k}. \tag{3.5}$$

Differentiation (3.3) on s yields

$$\mathrm{R}e\,[e^{-i\alpha_k}\psi(t)t'(s)] = 0, \quad |t-a_k| = r_k,\ k = 1, 2, \ldots, n. \tag{3.6}$$

Using (3.5) we arrive at the boundary value problem (3.1).

It follows from (3.2) that $\varphi(z)$ is analytic in D except at the infinite point where it satisfies the hydrodynamic normalization at infinity [21]

$$\varphi(z) = z + \varphi_0 + \frac{\varphi_1}{z} + \frac{\varphi_2}{z^2} + \ldots. \tag{3.7}$$

The function $\varphi(z)$ is multi-valued in D. More precisely, it is represented in the form [27]

$$\varphi(z) = z + \varphi_0(z) + \sum_{k=1}^{n} e^{i\alpha_k} A_k \ln(z-a_k), \tag{3.8}$$

where $\varphi_0(z)$ is single-valued analytic in D and A_k are undetermined real constants. The logarithm $\ln(z-a_k)$ is defined in such a way that it is analytic in the complex plane except a cut connecting the points $z = a_k$ and infinity. It is assumed that the cut does not cross $|z-a_m| \le r_m$ for $m \neq k$. The term $e^{i\alpha_k}A_k$ has such a form since the increment of the function $\mathrm{R}e\,[e^{-i\alpha_k}\varphi(t)]$ along $|t-a_k| = r_k$ must vanish because of (3.3). The problem (3.3) is discussed for multi-valued functions as well as for single-valued functions when all $A_k = 0$.

We use the Banach space $\mathcal{H}^{\mu}(L)$ consisting of functions Hölder continuous on Lyapunov's curve L endowed the norm

$$||\omega|| = \sup_{t\in L} |\omega(t)| + \sup_{t_{1,2}\in L} \frac{|\omega(t_1)| - \omega(t_2)|}{|t_1-t_2|^{\mu}}, \tag{3.9}$$

where $0 < \mu \le 1$. Analytic functions considered in the present paper can be continuous or continuously differentiable in the closures of the analyticity domains. The space $\mathcal{H}^{(k,\mu)}(L)$ consists of those functions which have Hölder continuous derivative of the kth order belonging to $\mathcal{H}^\mu(L)$. Let $\partial\Omega$ be the boundary of a domain Ω not necessary connected. Introduce a space $\mathcal{H}_A^\mu(\Omega)$ consisting of functions analytic in Ω and Hölder continuous in the closure of Ω endowed the norm (3.9). The space $\mathcal{H}_A^\mu(\Omega)$ is Banach, since the maximum principle for analytic functions implies that the norm in $\mathcal{H}_A^\mu(\Omega)$ coincides with the norm in $\mathcal{H}^\mu(\partial\Omega)$. One can consider $\mathcal{H}_A^\mu(\Omega)$ as a closed subspace of $\mathcal{H}^\mu(\partial\Omega)$. The space $\mathcal{H}_A^{(k,\mu)}(\Omega)$ is introduced in the same way as a subspace of $\mathcal{H}^{(k,\mu)}(\Omega)$. Therefore, the boundary value problems (3.1) and (3.3) are considered in the spaces $\mathcal{H}_A^{(\mu)}(D)$ and $\mathcal{H}_A^{(1,\mu)}(D)$, respectively.

Lemma 3.1 ([12])**.** *The problem* (3.3), (3.7) *for single-valued functions has a unique solution up to an arbitrary additive constant.*

Let $\zeta = u+iv$ denotes a complex variable on the complex plane with slits Γ_k $(k = 1, 2, \dots, n)$ lying on the lines

$$-\sin\alpha_k\, u + \cos\alpha_k\, v = c_k, \tag{3.10}$$

where c_k are the same as in (3.1). Let D' denote the complement of all the segments Γ_k to $\widehat{\mathbb{C}}$. The conformal mapping $\widetilde{\varphi}(z) = u(z) + iv(z)$ from D onto D' satisfies the boundary value problem (3.3), (3.7). It follows from Lemma 3.1 that the conformal mapping $\widetilde{\varphi}(z)$ coincides with the unique solution $\varphi(z)$ of the problem (3.3), (3.7) up to an additive constant.

Lemma 3.2. *The problem* (3.3), (3.7) *for multi-valued functions represented in the form* (3.8) *has* $(n + 1)$ $\mathbb{R}$*–linear independent solutions.*

Proof. One independent solution is a constant and other n independent solutions are produced by the terms $e^{i\alpha_k} A_k \ln(z - a_k)$ in the representation (3.8). Another proof follows from the relation between the problems (3.3), (3.7) and (3.1), (3.2). The winding number (index) of the problem (3.1), (3.2) is equal to n. Hence, it has n $\mathbb{R}$–linear independent solutions. The $(n + 1)$th solution is a constant obtained by integration of (3.1).

Remark 3.3. According to [18] the winding number $\varkappa$ of the problem (3.1) is equal to $(n + 1)$. The number of $\mathbb{R}$–linear independent solutions is equal to $\varkappa$ and the inhomogeneous problem corresponding to (3.1) is always solvable. The condition (3.2) reduces the number of $\mathbb{R}$–linear independent solutions to n that is in agree with the above conclusion.

The problem (3.3) for multi–valued functions can be reduced to the $\mathbb{R}$–linear problem [28]

$$\begin{aligned}\varphi(t) = \varphi_k(t) - e^{2i\alpha_k}\overline{\varphi_k(t)} + e^{i\alpha_k} c_k + e^{i\alpha_k}\xi_k \ln \tfrac{t-a_k}{r_k},\\ |t - a_k| = r_k,\ k = 1, 2, \dots, n,\end{aligned} \tag{3.11}$$

where $\varphi_k(z)$ is analytic in $|z-a_k|<r_k$ and continuously differentiable in $|z-a_k|\leq r_k$, real constants ξ_k are undetermined.

Lemma 3.4. *(i) Let $\varphi(z)$ and $\varphi_k(z)$ be solutions of* (3.11) *with arbitrarily fixed real constants ξ_k. Then $\varphi(z)$ satisfies* (3.3).

(ii) Let $\varphi(z)$ be a solution of (3.3) *and real constants ξ_k are arbitrarily fixed. Then there exist such functions $\varphi_k(z)$ that for each $k=1,\ldots,n$ the $\mathbb{R}$–linear conditions* (3.11) *are fulfilled.*

Proof of the first assertion is evident. It is sufficient to multiply (3.11) by $e^{-i\alpha_k}$ and to take the real part.

Conversely, let $\varphi(z)$ satisfies (3.3) and a real constant ξ_k is fixed. The function $e^{-i\alpha_k}\varphi_k(z)$ can be uniquely determined up to an additive real constant from the simple Schwarz problem for the disk $|z-a_k|<r_k$ [18], [27]

$$2\mathrm{I}m\ \left[e^{-i\alpha_k}\varphi_k(t)\right]=\mathrm{I}m\ \left[e^{-i\alpha_k}\varphi(t)-\xi_k\ln\frac{t-a_k}{r_k}\right],\quad |t-a_k|=r_k. \quad (3.12)$$

It follows from the later boundary condition that the function $\varphi_k(z)$ belongs to the spaces $\mathcal{H}_A^{(1,\alpha)}(D)$ except at the point $z=a_k$ where $\mathrm{I}m\ \ln(t-a_k)=\arg(t-a_k)$ has a discontinuity.

The lemma is proved.

Differentiate (3.11) on s along the circles $|t-a_k|=r_k$ and divide the results by $t'(s)$ calculated with (3.5)

$$\psi(t)=\psi_k(t)+e^{2i\alpha_k}\left(\frac{r_k}{t-a_k}\right)^2\overline{\psi_k(t)}+\frac{e^{i\alpha_k}\xi_k}{t-a_k},\quad |t-a_k|=r_k,\ k=1,2,\ldots,n, \quad (3.13)$$

where $\psi(z)=\varphi'(z)$ and $\psi_k(z)=\varphi_k'(z)$. Therefore, the Riemann–Hilbert problem (3.1) is reduced to the $\mathbb{R}$–linear problem (3.13).

4 Functional equations

The $\mathbb{R}$–linear problem (3.13) can be reduced to functional equations. Following [27, 28] introduce the function

$$\Phi(z):=\begin{cases}\psi_k(z)-\sum_{m\neq k}e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z^*_{(m)}\right)}-\sum_{m\neq k}\frac{e^{i\alpha_m}\xi_m}{z-a_m}, \\ \qquad\qquad |z-a_k|\leq r_k,\quad k=1,2,\ldots,n, \\ \psi(z)-\sum_{m=1}^{n}e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\left(z^*_{(m)}\right)}-\sum_{m=1}^{n}\frac{e^{i\alpha_m}\xi_m}{z-a_m},\ z\in D\end{cases}$$

analytic in $|z-a_k|<r_k$ $(k=1,2,\ldots,n)$ and D. Calculate the jump across the circle $|t-a_k|=r_k$

$$\Delta_k := \Phi^+(t) - \Phi^-(t),\ |t-a_k| = r_k,$$

where $\Phi^+(t) := \lim_{z\to t\ z\in D} \Phi(z)$, $\Phi^-(t) := \lim_{z\to t\ z\in D_k} \Phi(z)$. Application of (3.13) gives $\Delta_k = 0$. It follows from the principle of analytic continuation that $\Phi(z)$ is analytic in the extended complex plane. Moreover, $\psi(\infty) = 1$ yields $\Phi(\infty) = 1$. Then Liouville's theorem implies that $\Phi(z) \equiv 1$. The definition of $\Phi(z)$ in $|z-a_k| \le r_k$ yields the following system of functional equations

$$\begin{aligned} \psi_k(z) = \sum_{m\neq k} e^{2i\alpha_m} \left(\tfrac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} + 1 + \textstyle\sum_{m\neq k} \tfrac{e^{i\alpha_m}\xi_m}{z-a_m}, \\ |z-a_k| \le r_k,\ k=1,2,\ldots,n. \end{aligned} \tag{4.1}$$

Let $\psi_k(z)$ $(k=1,2,\ldots,n)$ be a solution of (4.1). Then the function $\psi(z)$ can be found from the definition of $\Phi(z)$ in D

$$\psi(z) = \sum_{m=1}^{n} e^{2i\alpha_m} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} + 1 + \sum_{m=1}^{n} \frac{e^{i\alpha_m}\xi_m}{z-a_m},\ z \in D \cup \partial D. \tag{4.2}$$

Consider inhomogeneous functional equations with any given element $f \in \mathcal{H}_A(\cup_{k=1}^n D_k)$

$$\psi_k(z) = \sum_{m\neq k} e^{2i\alpha_m} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} + f(z), |z-a_k| \le r_k,\ k=1,2,\ldots,n. \tag{4.3}$$

Theorem 4.1 ([27]). *The system* (4.3) *has a unique solution for any circular multiply connected domain D. This solution can be found by the method of successive approximations convergent in the space $\mathcal{H}_A(\cup_{k=1}^n D_k)$, i.e., uniformly convergent in every disk $|z-a_k| \le r_k$.*

The system of functional equations (4.1) can be decomposed onto $(n+1)$ systems

$$\psi_k^{(1)}(z) = \sum_{m\neq k} e^{2i\alpha_m} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m^{(1)}\left(z^*_{(m)}\right)} + 1, |z-a_k| \le r_k,\ k=1,2,\ldots,n. \tag{4.4}$$

and

$$\begin{aligned} \Psi_k^{(\ell)}(z) = \sum_{m\neq k} e^{2i\alpha_m} \left(\tfrac{r_m}{z-a_m}\right)^2 \overline{\Psi_m^{(\ell)}\left(z^*_{(m)}\right)} + \tfrac{e^{i\alpha_\ell}}{z-a_\ell}\, \delta'_{\ell k}, |z-a_k| \le r_k, \\ k=1,2,\ldots,n, \end{aligned} \tag{4.5}$$

where $\delta'_{\ell k} = 1 - \delta_{\ell k}$ and $\delta_{\ell k}$ is the Kronecker symbol. The unique solution of (4.1) can be represented in the form

$$\psi_k(z) = \psi_k^{(1)}(z) + \sum_{\ell=1}^{n} \xi_\ell \ \Psi_k^{(\ell)}(z). \tag{4.6}$$

The functions $\psi_k(z)$ can be constructed by two methods. First, they can be constructed by iterations applied to (4.1); second, by iterations applied separately to (4.4) and to (4.5) and further their linear combination (4.6). For any fixed $\psi_k(z)$, these iterations yield a series (in general conditionally convergent) with two different orders of summations. It follows from Theorem 4.1 that the result will be the same since we construct the same unique solution of (4.1) by two different methods. It is worth noting that (4.6) is a $\mathbb{C}$–linear combination of the basic functions because $\xi_\ell \in \mathbb{R}$ for $\ell = 1, 2, \dots, n$.

We now apply Theorem 4.1 to (4.4). Let $w \in D$ be a fixed point not equal to infinity. Introduce the functions

$$\phi_m(z) = \int_{w^*_{(m)}}^{z} \psi_m^{(1)}(t)dt + \phi_m(w^*_{(m)}), \quad |z - a_m| \le r_m, \quad m = 1, 2, \dots, n, \tag{4.7}$$

and

$$\omega(z) = -\sum_{m=1}^{n} e^{2i\alpha_m} \left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)} \right]. \tag{4.8}$$

The functions $\omega(z)$ and $\phi_m(z)$ analytic in D and in D_m, respectively, and continuously differentiable in the closures of the domains considered. One can see from (4.7) that the function $\phi_m(z)$ is determined by $\psi_m(z)$ up to an additive constant which vanishes in (4.8). The function $\omega(z)$ vanishes at $z = w$. Investigate the function $\omega(z)$ on the boundary of D. It follows from (4.8) and $t = t^*_{(k)}$ ($|t - a_k| = r_k$) for each fixed k that

$$\omega(t) = -e^{2i\alpha_k} \left[\overline{\phi_k(t)} - \overline{\phi_k\left(w^*_{(k)}\right)} \right] - \Psi_k(t), \tag{4.9}$$

where

$$\Psi_k(z) = \sum_{m \ne k} e^{2i\alpha_m} \left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)} \right]. \tag{4.10}$$

Using the relation [27]

$$\frac{d}{dz} \left[\overline{\phi_m\left(z^*_{(m)}\right)} \right] = -\left(\frac{r_m}{z - a_m} \right)^2 \overline{\phi'_m\left(z^*_{(m)}\right)}, \ |z - a_m| > r_m, \tag{4.11}$$

calculate the derivative

$$\Psi_k'(z) = -\sum_{m=1}^{n} e^{2i\alpha_m} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m^{(1)} \left(z_{(m)}^* \right)}. \tag{4.12}$$

Application of (4.1) yields

$$\Psi_k'(z) = 1 - \psi_k^{(1)}(z). \tag{4.13}$$

Then (4.9) and (4.7) implies that

$$e^{-i\alpha_k}\omega(t) = -e^{i\alpha_k} \left[\overline{\phi_k(t)} - \overline{\phi_k \left(w_{(k)}^* \right)} \right] + e^{-i\alpha_k}[\phi_k(t) - t + d_k], \quad |t - a_k| = r_k, \tag{4.14}$$

where d_k is a constant of integration. Calculation of the real part of (4.14) gives

$$\mathrm{Re}\, [e^{-i\alpha_k}(\omega(t) + t)] = p_k, \quad |t - a_k| = r_k, \tag{4.15}$$

where p_k is a constant. Comparing (4.15) with (3.3) and using Lemma 3.1 we conclude that the conformal mapping D onto D' has the form

$$\widetilde{\varphi}(z) = z + \omega(z) + constant, \tag{4.16}$$

where $\omega(z)$ is calculated by (4.8).

Application of the method of successive approximations to (4.4) and integration terms by terms of the obtained uniformly convergent series yields the exact formula

$$\varphi_k(z) = q_k + z - \sum_{k_1 \neq k} e^{2i\alpha_{k_1}} (\overline{z_{(k_1)}^* - w_{(k_1)}^*}) + \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} e^{2i(\alpha_{k_1} - \alpha_{k_2})} (z_{(k_2 k_1)}^* - w_{(k_2 k_1)}^*) - \tag{4.17}$$

$$\sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \sum_{k_3 \neq k_2} e^{2i(\alpha_{k_1} - \alpha_{k_2} + \alpha_{k_3})} (\overline{z_{(k_3 k_2 k_1)}^* - w_{(k_3 k_2 k_1)}^*}) + \ldots, \quad |z - a_k| \leq r_k.$$

Using (4.8) and (4.17) we write the function (4.16) up to an arbitrary additive constant in the form

$$\widetilde{\varphi}(z) = z - \sum_{k=1}^{n} e^{2i\alpha_k} (\overline{z_{(k)}^* - w_{(k)}^*}) + \sum_{k=1}^{n} \sum_{k_1 \neq k} e^{2i(\alpha_k - \alpha_{k_1})} (z_{(k_1 k)}^* - w_{(k_1 k)}^*) - \tag{4.18}$$

$$\sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} e^{2i(\alpha_k - \alpha_{k_1} + \alpha_{k_2})} (\overline{z_{(k_2 k_1 k)}^* - w_{(k_2 k_1 k)}^*}) + \ldots$$

Differentiation of the latter uniformly convergent series term by term yields the $\boldsymbol{\alpha}$– series (2.8) with $H(z) = 1$

$$\psi^{(1)}(z) = 1 - \sum_{k=1}^{n} e^{2i\alpha_k} (\overline{z^*_{(k)}})' + \sum_{k=1}^{n} \sum_{k_1 \neq k} e^{2i(\alpha_k - \alpha_{k_1})} (z^*_{(k_1 k)})' - \tag{4.19}$$

$$\sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} e^{2i(\alpha_k - \alpha_{k_1} + \alpha_{k_2})} (\overline{z^*_{(k_2 k_1 k)}})' + \dots$$

A similar method can be used to construct $\Psi_k^{(\ell)}(z)$ satisfying (4.5) and to construct

$$\Psi^{(\ell)}(z) = \sum_{m=1}^{n} e^{2i\alpha_m} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\Psi_m^{(\ell)} \left(z^*_{(m)} \right)} + \frac{e^{i\alpha_\ell}}{z - a_\ell}, \ z \in D \cup \partial D. \tag{4.20}$$

We have

$$\Psi^{(\ell)}(z) = \frac{e^{i\alpha_\ell}}{z - a_\ell} - e^{-i\alpha_\ell} \sum_{k=1}^{n} \frac{e^{2i\alpha_k}}{\overline{z^*_{(k)} - a_k}} (\overline{z^*_{(k)}})' + e^{i\alpha_\ell} \sum_{k=1}^{n} \sum_{k_1 \neq k} \frac{e^{2i(\alpha_k - \alpha_{k_1})}}{z^*_{(k_1 k)} - a_{k_1}} (z^*_{(k_1 k)})' +$$
$$- e^{-i\alpha_\ell} \sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \frac{e^{2i(\alpha_k - \alpha_{k_1} + \alpha_{k_2})}}{\overline{z^*_{(k_2 k_1 k)} - a_{k_2}}} (\overline{z^*_{(k_2 k_1 k)}})' + \dots \tag{4.21}$$

Therefore, the general solution of the Riemann–Hilbert problem (3.1) has the form

$$\psi(z) = \psi^{(1)}(z) + \sum_{\ell=1}^{n} \xi_\ell \ \Psi^{(\ell)}(z), \tag{4.22}$$

where $\psi^{(1)}(z)$ is given by (4.19) and $\Psi^{(\ell)}(z)$ by (4.21).

Integration of (4.22) from w to z yields

$$\varphi(z) = \widetilde{\varphi}(z) + \sum_{\ell=1}^{n} \xi_\ell \ \widetilde{\varphi}^{(\ell)}(z) + constant, \tag{4.23}$$

where $\widetilde{\varphi}(z)$ has the form (4.18). The function $\widetilde{\varphi}^{(\ell)}(z)$ is written explicitly

$$\widetilde{\varphi}^{(\ell)}(z) = e^{i\alpha_\ell} \ln \frac{z - a_\ell}{w - a_\ell} - e^{-i\alpha_\ell} \sum_{k=1}^{n} e^{2i\alpha_k} \ln \frac{\overline{z^*_{(k)} - a_k}}{\overline{w^*_{(k)} - a_k}} \tag{4.24}$$

$$+ e^{i\alpha_\ell} \sum_{k=1}^{n} \sum_{k_1 \neq k} e^{2i(\alpha_k - \alpha_{k_1})} \ln \frac{z^*_{(k_1 k)} - a_{k_1}}{z^*_{(k_1 k)} - a_{k_1}}$$

$$-e^{-i\alpha_\ell}\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1} e^{2i(\alpha_k-\alpha_{k_1}+\alpha_{k_2})}\ln\frac{\overline{z^*_{(k_2k_1k)}-a_{k_2}}}{\overline{w^*_{(k_2k_1k)}-a_{k_2}}}+\ldots$$

It is worth noting that that separation of the terms with z and w in (4.24) can fail to converge [28].

In order to compare (4.23) and (3.8) we note that the conformal mapping $\widetilde{\varphi}(z)$ coincides with $z+\varphi_0(z)$ up to an additive constant. Hence,

$$\sum_{k=1}^{n} e^{i\alpha_k} A_k \ln(z-a_k)=\sum_{\ell=1}^{n}\xi_\ell\,\widetilde{\varphi}^{(\ell)}(z),\quad z\in D. \tag{4.25}$$

Substitution of (4.24) into (4.25) can yield relations between the constants A_k and ξ_ℓ.

5 Schottky double

The Schottky double $\mathcal{S}$ is obtained from two equal multiply connected domains D and $\widetilde{D}\equiv D$ glued along the circles $|t-a_k|=r_k$ $(k=1,2,\ldots,n)$. Analytic functions in a domain of $\mathcal{S}$ are those functions which are analytic on $D\cup\partial D$ in z and analytic on $\widetilde{D}\cup\partial\widetilde{D}$ in $\overline{z}$ with the condition $\Phi(t)=\widetilde{\Phi}(\overline{t})$ on the joint part of ∂D. Hence, the Schottky double $\mathcal{S}$ is a compact Riemann surface of genus $(n-1)$ [40]. Let t_k be a fixed point on the circle $|t-a_k|=r_k$ and $\mathbf{a}'\subset D$ be a simple smooth curve connecting the points t_n and t_k $(k=1,2,\ldots,n-1)$. Introduce the symmetric curve $\widetilde{\mathbf{a}'}_k\subset\widetilde{D}$ connecting the points t_k and t_n and the closed curve $\mathbf{a}_k=\mathbf{a}'\cup\widetilde{\mathbf{a}'}_k$ on $\mathcal{S}$. Let $\mathbf{b}_k$ denote the clockwise oriented circle $|t-a_k|=r_k$. The curves $\mathbf{a}_k$ and $\mathbf{b}_k$ $(k=1,2,\ldots,n-1)$ form a homology basis for $\mathcal{S}$ and any cycle on $\mathcal{S}$ is homologous to a linear combination of $\mathbf{a}_k$ and $\mathbf{b}_k$ with integer coefficients.

The harmonic measure $\omega_\ell(z)$ of the circle $|t-a_\ell|=r_\ell$ relative to the multiply connected domain D is a function harmonic in D continuous in $D\cup\partial D$ which satisfies the Dirichlet problem

$$\omega_\ell(t)=\delta_{\ell k},\quad |t-a_k|=r_k\ (k=1,2,\ldots,n), \tag{5.1}$$

where $\delta_{\ell k}$ stands for the Kronecker symbol. The harmonic measures were constructed in [23, 26, 27] in terms of the Poincaré θ_2–series (2.3). Let $\widetilde{\omega}_\ell(z)$ be a multi–valued function harmonically conjugated to $\omega_\ell(z)$. The functions $w_\ell(z)=\omega_\ell(z)+i\widetilde{\omega}_\ell(z)$ $(\ell=1,2,\ldots,n-1)$ analytic in D are called the normalized Abelian integrals of first kind in D. The differentials $dw_\ell(z)$ generate the linear space of the Abelian differentials of first kind and $dw_\ell(z)$ $(\ell=1,2,\ldots,n-1)$ form the basis of this space. Each differential $dw_\ell(z)$

takes pure imaginary values on ∂D. Hence, it can be analytically continued into $\widetilde{D}$ in the topology of the Schottky double by the symmetry principle. Moreover, $dw_\ell(z)$ is single–valued on $\mathcal{S}$.

The periods of the Abelian differentials

$$\int_{\mathbf{a}_k} dw_m(t) = 2\int_{\mathbf{a}'_k} dw_m(t), \quad B_{km} = \int_{\mathbf{b}_k} dw_m(t) \quad (k = 1, 2, \ldots, n-1)$$

form two matrix. The second one has the form iB, where $B = \{B_{km}\}$ is a real negatively determined matrix. Following [40] we consider the real Jacobi inversion problem. Let $w_k^-(t)$ denote the limit values of the Abelian integral on the curve $\mathbf{a}'_k$ when z tends to $t \in \mathbf{a}'_k$ from the right side of the curve $\mathbf{a}'_k$. The function $w_k^-(t)$ is multi–valued. We fix any its branch in the simply connected domain $D\backslash \left(\cup_{k=1}^{n-1}\mathbf{a}'_k\right)$ where it is single–valued. Given constants e_k $(k = 1, 2, \ldots, n-1)$. To find the points z_m $(m = 1, 2, \ldots, n-1)$ in $D \cup \partial D$ satisfying the relation

$$\sum_{m=1}^{n-1} \text{Im } w_k(z_m) \equiv e_k - \frac{1}{2}B_{kk} + \sum_{m\neq k}^{n-1} \text{Im} \int_{\mathbf{a}'_k} w_k^-(t) dw_m(t), \; (k = 1, 2, \ldots, n-1). \tag{5.2}$$

Here, $\equiv$ means equality modulo B–periods. The generalized real Jacobi inversion problem has the form [40]

$$\sum_{m=1}^{n-1} \text{Im } w_k(z_m) \equiv \frac{1}{2\pi i}\int_{\partial D} \gamma(t) dw_m(t), \; (k = 1, 2, \ldots, n-1), \tag{5.3}$$

where $\gamma(t)$ is a given Hölder continuous function except at a finite number of points where finite step discontinuities are possible.

Let $\lambda(t)$ be a given Hölder continuous function on ∂D satisfying the condition $|\lambda(t)| = 1$. The Riemann–Hilbert problem

$$\text{Im } [\overline{\lambda(t)}\psi(t)] = 0, \quad t \in \partial D, \tag{5.4}$$

was solved in terms of the $\boldsymbol{\alpha}$–series [23, 26, 27]. Let the functions $\lambda(t)$ from (5.4) and $\gamma(t)$ from (5.3) are related by formula

$$\lambda(t) = \exp[i\gamma(t)]. \tag{5.5}$$

We now consider the particular case (3.1) of the problem (5.4) and the corresponding generalized Jacobi inversion problem (5.3). We have

$$\lambda(t) = \frac{e^{i\alpha_k} r_k}{t - a_k}, \; \gamma(t) = \alpha_k - \arg\frac{t - a_k}{r_k}, \; |t - a_k| = r_k, \; (k = 1, 2, \ldots, n). \tag{5.6}$$

The branch of the argument corresponds to the chosen branch of the logarithm $\ln(t-a_k)$ from Sec.3. Each non–trivial solution of the problem (3.1) has exactly $n-1$ zeros z_m $(m=1,2,\ldots,n-1)$ in $D\cup\partial D$ which solve the generalized Jacobi inversion problem (5.3) with $\gamma(t)$ given by (5.6). The $\boldsymbol{\alpha}$–series (4.19) is a solution of (3.1).

The conditions $\xi_k=0$ by (4.25) implies that all $A_k=0$ in the representation (3.8). Hence, this case corresponds to the problem (3.3), (3.7) in a class of single–valued functions. The unique solution of this problem is given by (4.18). This function is the conformal mapping of the domain D onto the slit domain D' with the normalisation (3.7). The function $\psi^{(1)}(z)$ given by (4.19) is the derivative of this conformal mapping. Hence, it cannot have zeros in the domain D. Therefore, all the zeros z_m $(m=1,2,\ldots,n-1)$ of $\psi(z)$ which solve the generalized Jacobi inversion problem (5.3), lie on the boundary ∂D. This observation can be useful to numerical solution of the Jacobi inversion problem on the Schottky double.

6 Fast algorithm

Though the complete solution of the Riemann–Hilbert problem for an arbitrary circular multiply connected domain was written explicitly in terms of the $\boldsymbol{\alpha}$–series, many mathematicians apply the standard absolutely convergent scheme to the Poincaré series and use direct methods of computation to the Poincaré series [36]. Perhaps, it is related to the fact that even absolutely convergent Poincaré series are slowly convergent for closely spaced disks. We suppose that modifications of the iterative functional equations can increase the convergence. In the present section, we discuss such a modification proposed in [32] to construct a basic solution of the problem (3.1). For brevity, we consider the classical Poincaré series when $\alpha_k=0$ $(k=1,2,3)$ for three equal disks $(r_k=r)$.

Consider an auxiliary problem for two disks. Let the domain G be the complement of two disjoint disks $|z-a_k|\leq r$ $(k=1,2)$ to the extended complex plane. The quadratic equation $z^*_{(1)}=z^*_{(2)}$ with respect to z has two roots

$$z_{12}=\frac{a_1+a_2}{2}-\frac{a_2-a_1}{2}\sqrt{1-2\frac{r_1^2+r_2^2}{|a_2-a_1|^2}},\tag{6.1}$$

$$z_{21}=\frac{a_1+a_2}{2}+\frac{a_2-a_1}{2}\sqrt{1-2\frac{r_1^2+r_2^2}{|a_2-a_1|^2}},$$

The complex potential

$$\Psi_{12}(z) = \frac{1}{z - z_{12}} - \frac{1}{z - z_{21}} \tag{6.2}$$

describes the flux between the disks when the difference $u_1 - u_2$ of the potentials on the boundaries of the disks is equal to

$$u_1 - u_2 = \ln \frac{1 - \sqrt{1 - \frac{4r^2}{|a_2 - a_1|^2}}}{1 + \sqrt{1 - \frac{4r^2}{|a_2 - a_1|^2}}}. \tag{6.3}$$

The main idea of the fast method is based on the decomposition of the complex flux $\psi(z)$ onto $\psi_\delta(z)$ and $\psi_0(z)$ where the singular function $\psi_\delta(z)$ has the form

$$\psi_\delta(z) = \Psi_{12}(z) + \Psi_{13}(z), \tag{6.4}$$

where $\Psi_{13}(z)$ is introduced similar to (6.2) (the subscript 2 is replaced by 3). A solution of the boundary value problem (3.1) for $n = 3$ is looked for in the form

$$\psi(z) = \psi_0(z) + \psi_\delta(z), \quad z \in \mathbb{D}, \tag{6.5}$$

where $\psi_\delta(z)$ is given by (6.4). The boundary value problem (3.1) becomes

$$\mathrm{Im}\, \frac{t - a_k}{r} \left[\psi_0(t) + \psi_\delta(t)\right] = 0, \quad |t - a_k| = r, \ k = 1, 2, 3. \tag{6.6}$$

Introduce the functions analytic in $|z - a_k| < r$

$$f_k(z) = \begin{cases} 0 & \text{for } k = 1 \\ \Psi_{13}(z) & \text{for } k = 2 \\ \Psi_{12}(z) & \text{for } k = 3. \end{cases} \tag{6.7}$$

One can see that

$$\mathrm{Im}\, \frac{t - a_m}{r} \Psi_{12}(t) = 0, \quad |t - a_m| = r \ (m = 1, 2)$$

and

$$\mathrm{Im}\, \frac{t - a_m}{r} \Psi_{13}(t) = 0, \quad |t - a_m| = r \ (m = 1, 3).$$

Then (6.6) can be written in the form

$$\mathrm{Im}\, \frac{t - a_k}{r} \left[\psi_0(t) + f_k(t)\right] = 0, \quad |t - a_k| = r, \ k = 1, 2, 3. \tag{6.8}$$

The boundary value problem (6.8) is reduced to the $\mathbb{R}$–linear problem

$$\psi_0(t) = \psi_k(t) + \left(\frac{r}{t-a_k}\right)^2 \overline{\psi_k(t)} - f_k(t),\ |t-a_k| = r,\ k = 1,2,3. \tag{6.9}$$

The problem (6.9) can be reduced to the following system of functional equations [27]

$$\psi_k(z) = \sum_{m\neq k} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} + f_k(z). \tag{6.10}$$

The iteration method can be applied to solve the system (4.1) [27]

$$\psi_k^{(0)}(z) = f_k(z), \tag{6.11}$$

$$\psi_k^{(p)}(z) = \sum_{m\neq k} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m^{(p-1)}\left(z^*_{(m)}\right)} + f_k(z),\ p = 1,2,\dots. \tag{6.12}$$

The p-th approximation of the complex flux is calculated by formula

$$\psi^{(p)}(z) = \sum_{m=1,2,3} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m^{(p)}\left(z^*_{(m)}\right)} + \psi_\delta(z), \quad z\in\mathbb{D}, \tag{6.13}$$

where $\psi_\delta(z)$ is given by (6.4).

Fig.2 describes the flux around three closely placed disks. It follows from computations that 6 iterations is sufficient to obtain an accessible result (the respective error on the boundary is 2%).

7 Discussion

In the present paper, we introduce the Poincaré $\boldsymbol{\alpha}$–series. First time, the $\boldsymbol{\alpha}$–series were used in [23] without their deep discussion to solve Riemann–Hilbert problems for an arbitrary circular multiply connected domain. The main difference in the methods [39] and [23, 30] applied to Riemann–Hilbert problems is that [39] is based on the Jacobi inversion problem and [23, 30] is not. But [23, 30] includes the $\boldsymbol{\alpha}$–series that coincide with the theta-function of Riemann. Hence, the solution of the Jacobi inversion problem is ultimately constructed in terms of the $\boldsymbol{\alpha}$–series. It is worth noting that solution to Riemann–Hilbert problems in [39] and later investigations by this scheme are not completed. First of all, it was assumed in [39] that the Abelian integrals of first kind were known. Substitution of the Abelian integrals into the multidimensional theta–series yielded the theta–function of Riemann.

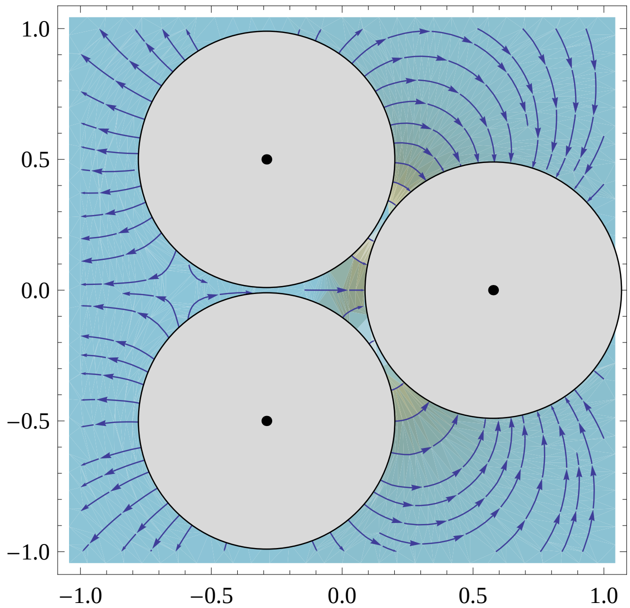

Figure 2: Streamlines of the complex flux $\psi(z)$ computed in the sixth order approximation around three disks with the centres at $a_1 = \frac{1}{\sqrt{3}}$, $a_2 = \frac{1}{\sqrt{3}}\, e^{\frac{2}{3}\pi i}$, $a_3 = \frac{1}{\sqrt{3}}\, e^{-\frac{2}{3}\pi i}$ of the radius 0.49.

The latter function was applied to investigate the Jacobi inversion problem. After this the Schwarz operator was applied to get the solution of the Riemann–Hilbert problem. Construction of the Abelian integrals (harmonic measures) and the Schwarz operator in terms of the classical θ_2–series of Poincaré [23, 30] could make this complicated scheme effective. However, the Jacobi inversion problem cannot be avoided in the scheme [39] .

Application of the $\boldsymbol{\alpha}$–series simplifies solution to Riemann–Hilbert problems by elimination of the Jacobi inversion problem and produces directly the theta–function of Riemann. Moreover, the scheme [23, 30] allows to constructively solve the Jacobi inversion problem as a separately stated problem. Bojarski's linear algebraic system (see Bojarski's addition to [38]) which describes solvability of the Riemann–Hilbert problem is explicitly written in terms of the $\boldsymbol{\alpha}$–series. The constructive method [23, 30] is valid for the Schottky double. It is interesting to extend it to the general compact Riemann surfaces.

Crowdy [8] stated open problems of the constructive theory of functions in multiply connected domains. In particular, Crowdy wrote [8] about Schwarz–Christoffel type conformal mappings: "The history of this particular problem also presents a paradigm for a key message of this paper: that, given modern advances in computational capability and in light of modern applications, many topics in classical geometric function theory can (and should!) be revisited and reappraised". I think that this phrase should concern the whole constructive theory of functions in multiply connected domains. In this paper, we answer some questions stated by Crowdy [8]. It is worth noting that these answers are not complete and require further investigations.

Question 1 of Crowdy addressed to the infinite product representation (2.19) which is always uniformly convergent.

Question 2 of Crowdy concerned effective computational methods. Such a fast method is presented in Sec.6.

Question 3 of Crowdy concerned the complicated scheme by Zverovich [39] used by many authors for Riemann-Hilbert problems. It is explained above in this section that the method [30] based on $\boldsymbol{\alpha}$–series is constructive and simpler than the method [39].

Crowdy in Question 4 paid attention to an alternative class of canonical multiply connected domains introduced by Bell [2, 3, 6, 35]. It can be add to this that Bell's domains have applications to neutral inclusions [19]. The latter problem is related to eigenvalue problems, Courant's nodal domain theorem and non–linear Riemann–Hilbert problems discussed in [33]. It is interesting to relate Bell's domains to eigenvalue problems for $\mathbb{R}$–linear conjugation condition [33, 34].

Question 5 of Crowdy addressed to the Riemann-Hilbert problem (3.3) and eventual use of the classical prime function. As it follows from the result of this paper, the $\boldsymbol{\alpha}$–prime function (2.20) can be applied to (3.3) and it is rather impossible to solve the problem (3.3) in terms of the classical prime function (2.19).

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