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$\mathbb{R}$-LINEAR PROBLEM FOR MULTIPLY CONNECTED DOMAINS AND ALTERNATING METHOD OF SCHWARZ

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Abstract. We study the $\mathbb{R}$-linear conjugation problem for multiply connected domains by the method of integral equations. The method differs from the classical method of potentials. It is related to the generalized alternating method of Schwarz, which is based on the decomposition of the considered domain with complex geometry into simple domains and subsequent solution to boundary value problems for simple domains. Convergence of the method of successive approximations is investigated.

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1. Introduction

The present paper is devoted to the $\mathbb{R}$-linear conjugation problem for multiply connected domains.

We develop a method of integral equations closely related to the generalized alternating method of Schwarz [12, 20], which is based on the decomposition of the considered domain with complex geometry into simple domains and subsequent solution of boundary-value problems for simple domains. In each step of the algorithm the boundary conditions for a simple domain are corrected by the influence of the other simple domains computed at the precedent steps. This method is referred to decomposition methods [32] frequently used in numerical computations and realized in the form of alternating methods.

There are various versions of the alternating methods in literature. The classical alternating method is applied to overlapping domains Ω_1 and Ω_2, when the solution to a boundary value problem for $\Omega_1 \cap \Omega_2$ or for $\Omega_1 \cup \Omega_2$ is constructed via a sequence of solutions to boundary value problems separately for Ω_1 and for Ω_2 (see Fig. 1(a)). Substructuring methods are addressed to domain decomposition methods when the overlap between the subdomains Ω_1 and Ω_2 is reduced to the interface (see Fig. 1(b)). The classical and substructuring methods always converge [12, 32].

The generalized alternating method of Schwarz is applied to nonoverlapping domains (see Fig. 1(c)). This method is effective in the study of composites, when nonoverlapping inclusions are embedded in a host material which occupies a domain Ω. Then the inclusions Ω_1 and Ω_2 interact with each other

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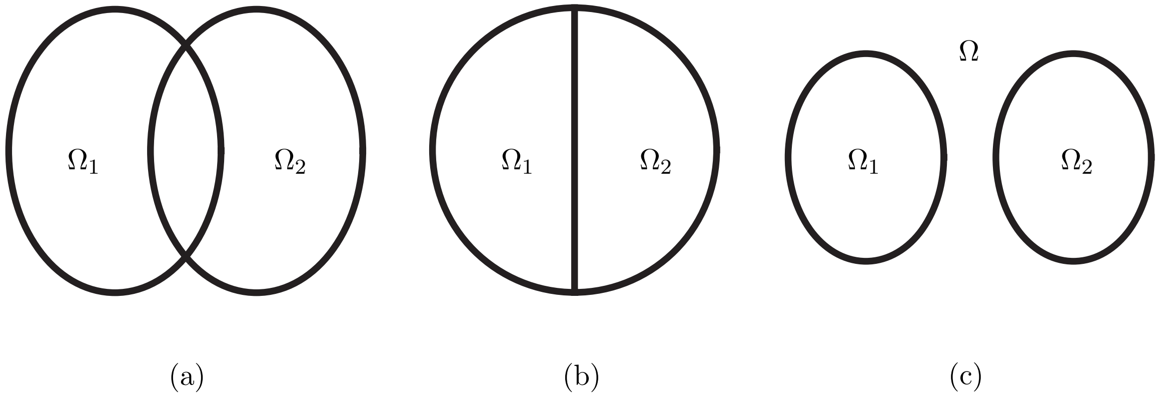

Fig. 1. Three cases of the geometry: (a) overlapping domains; (b) overlap reduced to the interface; (c) nonoverlapping domains.

through the host material. In the previous works, the absolute convergence of the method was proved under geometrical restrictions [20].

However, it was proved in [22, 27] for the case of circular multiply connected domains that a modified method always uniformly converges, i.e., these geometrical restrictions are redundant. This is an interesting example of the difference between absolute and uniform convergence which shows that estimations on the absolute values or on the norm are too strong in comparison to the study of the uniform convergence. This result leads to the conjecture that the generalized alternating method of Schwarz for boundary value problems with zero winding number (index) always uniformly converges.

In the present paper, the method of Schwarz is modified by the integral equation method proposed by Mikhajlov [19]. In the case of circular domains this method differs from the method of Schwarz by addition of the constant term [27]. But this slight modification yields a convergent algorithm. In the case of general domains, this modification corresponds to adding an integral term. This modification also yields the convergence of the method for the $\mathbb{R}$-linear problem with zero winding number. It is worth noting that the recent results [33] obtained for the Riemann–Hilbert problem for an annulus show that convergence fails for nonzero winding numbers.

The paper is organized as follows. In the next subsections, the $\mathbb{R}$-linear problem is stated in various forms and a survey devoted to the previous results is given. Section 2 is devoted to extension of [4] to multiply connected domains. In Sec. 3, integral equations corresponding to the generalized alternating method of Schwarz are deduced. It is proved that the method of successive approximations can be applied to these equations in Sec. 4.

1.1. Statements of the problems. Let D_k be mutually disjoint simply connected domains in the complex plane $\mathbb{C}$ bounded by smooth curves L_k ($k = 1, 2, \dots, n$), and D be the complement of all closures of D_k to the extended complex plane $\mathbb{C} \cup \{\infty\}$. Below, the domains D_k are called inclusions. Denote by D^+ the union of all inclusions D_k, i.e., the domain D^+ consists of n connected components. Let L_k be oriented in the counterclockwise direction. Let $a(t)$, $b(t)$, and $c(t)$ be given Hölder-continuous functions on the boundary $L = \bigcup_{k=1}^{n} L_k$ of D^+ and let $a(t)$ not vanish on L.

The $\mathbb{R}$-*linear conjugation problem* is stated as follows: Find a function $\varphi(z)$ that is analytic in D and in all components of D^+ and continuous in the closures of the considered domains with the following conjugation condition:

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + c(t), \quad t \in L. \tag{1}$$

where $\varphi^{\pm}(t)$ denote the limit values of $\varphi(z)$ as z tends to a point $t \in L$ from D^+ and from D, respectively. Moreover, $\varphi(z)$ vanishes at infinity.

The $\mathbb{R}$-linear problem with constant on each L_k coefficients $a(t)$, $b(t)$ is equivalent to the *transmission problem*

$$u^+(t) = u^-(t) + c_1(t), \tag{2}$$

$$\lambda_k \frac{\partial u^+}{\partial n}(t) = \lambda \frac{\partial u^-}{\partial n}(t) + c_2(t), \quad t \in L_k,\ k = 1, 2, \ldots, n, \tag{3}$$

where the real function $u(z)$ is harmonic in D and in all components of D^+ and continuously differentiable in the closures of the considered domains, and $\partial/\partial n$ denotes the normal derivative. The conjugation conditions (2)–(3) express the perfect contact between materials with different conductivities λ and λ_k (see [24]). Below, we demonstrate the equivalence of the problems (1) and (2)–(3) in the considered case.

Let $v(z)$ be a harmonic function conjugated to $u(z)$ which is determined up to an additive constant in each component of the complex plane. Using the Cauchy–Riemann equations on L_k [8], we can rewrite relation (3) in the form

$$\lambda_k \frac{\partial v^+}{\partial s}(t) = \lambda \frac{\partial v^-}{\partial s}(t) + c_2(t), \quad t \in L_k,\ k = 1, 2, \ldots, n, \tag{4}$$

where $\partial/\partial s$ denotes the tangent derivative along L_k. Integration of (4) yields up to additive constants

$$\lambda_k v^+(t) = \lambda v^-(t) + C_2(t), \quad t \in L_k,\ k = 1, 2, \ldots, n, \tag{5}$$

where $C_2(t) = \int c_2(t)ds$. Two real equations (2) and (5) are equivalent to one complex equality

$$\varphi^+(t) = \frac{\lambda_k + \lambda}{2\lambda_k}\varphi^-(t) + \frac{\lambda_k - \lambda}{2\lambda_k}\overline{\varphi^-(t)} + c(t), \quad t \in L_k, \tag{6}$$

where

$$\varphi(z) = u(z) + iv(z), \quad c(t) = c_1(t) + iC_2(t).$$

1.2. Bibliographic notes. In the case $b(t) = 0$, problem (1) becomes the $\mathbb{C}$-*linear conjugation problem*

$$\varphi^+(t) = a(t)\varphi^-(t) + c(t), \quad t \in \partial D. \tag{7}$$

Problem (7), the Riemann–Hilbert problem, and corresponding singular integral equations were systematically studied in the 20th Century by Gakhov [8], Muskhelishvili [29], Vekua [34], and by many other mathematicians, including their disciples (see, e.g., [26, 27]).

Now we briefly describe the history of the $\mathbb{R}$-linear problem (1). It is curious that for many years this problem were independently discussed as two different problems (1) and (2)–(3) by various mathematicians and also by the same mathematicians. In 1932, using the theory of potentials, Muskhelishvili [28] (see also [30, p. 522]) reduced problem (2)–(3) to a Fredholm integral equation and proved that it has a unique solution in the case of positive λ and λ_k, the most interesting in applications. In 1933, Vekua and Ruhadze [35, 36] constructed a solution of (2)–3 in closed form for an annulus and an ellipse (see also papers by Ruhadze quoted in [30]). Hence, [28] is the first result on the solvability of the $\mathbb{R}$-linear problem and [35, 36] are the first papers devoted to exact solutions of special cases of the $\mathbb{R}$-linear problem.

In 1946, Markushevich [17] again stated the $\mathbb{R}$-linear problem in the form (1) and studied it in the case $a(t) = 0$, $b(t) = 1$, and $c(t) = 0$ when (1) is not a Nöther problem. Later, Muskhelishvili [29] did not determined whether (1) was his problem (2)–(3) discussed in 1932 in terms of harmonic functions.

In 1960, Bojarski [4] showed that in the case $|b(t)| < |a(t)|$ with $a(t)$ and $b(t)$ belonging to the Hölder class $\mathcal{H}^{1-\varepsilon}$ with sufficiently small ε, the $\mathbb{R}$-linear problem (1) is qualitatively similar to the $\mathbb{C}$-linear problem (7) for simply connected domains. This result is precisely formulated in Theorem 1 in the next section and it is proved for multiply connected domains. Later, Mikhailov [19] (first published in 1961 in [18]) developed this result to continuous coefficients $a(t)$ and $b(t)$; $c(t) \in L_p(L)$. The case

$$|b(t)| < |a(t)| \tag{8}$$

was called the elliptic case. It corresponds to the partial case considered by Muskhelishvili [28]. Mikhajlov [19] reduced problem (1) to an integral equation and justified the absolute convergence of the method of successive approximation for the latter equation in the space $L_p(L)$ under the restrictions $\operatorname{wind}_L a(t) = 0$ and

$$(1 + S_p)|b(t)| < 2|a(t)|, \tag{9}$$

where S_p is the norm of the singular integral in $L_p(L)$ and $\operatorname{wind}_L a(t)$ is the winding number (index) of $a(t)$ along L.

In the case $|a(t)| \equiv |b(t)|$, problem (1) is reduced to the Riemann–Hilbert problem. One can find the solution of the latter problem for simply connected domains in [8, 29]. The Riemann–Hilbert problem for multiply connected domains was discussed in Bojarski's supplement to the book [34]. In the case $a(t) \equiv b(t) \equiv 1$, problem (1) is reduced to n Schwarz problems for the simply connected domains D_k $(k = 1, 2, \dots, n)$

$$\operatorname{Im} \varphi^+(t) = \operatorname{Im} c(t), \quad t \in L_k, \tag{10}$$

and to the problem for the multiply connected domain D

$$2 \operatorname{Re} \varphi^-(t) = \operatorname{Re}[c(t) - \varphi^+(t)], \quad t \in L. \tag{11}$$

It is worth noting that the Schwarz problem and the classic Dirichlet problems are equivalent for simply connected domains and closely related to each other for multiply connected domains [20].

Another independent study of the $\mathbb{R}$-linear problem (1) began in 1934 from Golusin's papers [9–11], where he reduced the Dirichlet problem for multiply connected circular domains to a system of functional equations and applied the method of successive approximations to obtain its solution under some geometrical restrictions. These restrictions correspond to the condition of absolute convergence of the generalized method of Schwarz (of the Poincaré series for circular domains [31]) and are equivalent to Mikhajlov's restriction (9). Golusin's approach was developed in [1, 6, 37].

In 1994–1998, the Riemann–Hilbert problem was explicitly solved [21, 23] for arbitrary multiply connected circular domain. This result were based first on the standard factorization method [8, 29], which allows one to reduce the Riemann–Hilbert problem to the Schwarz problem. The latter problem were reduced to functional equations, a modification of Golusin's equations. Then, the uniform convergence of the successive approximations for the functional equations was justified by using the result [4]. The series that arose actually coincided with the Poincaré series. This also gave the proof of the uniform convergence of the Poincaré series for any multiply connected circular domain [22].

Problems (1) and (2)–(3) with partial coefficients for special domains were considered by many mathematicians as related to applied problems of a continuum. One can find corresponding references in [16, 24, 26, 27]. Dzhuraev [7] and Komyak [13, 14] investigated a relation between the $\mathbb{R}$-linear problem and two-dimensional singular integral equations. Litvinchuk and Spitkovsky [16] studied the $\mathbb{R}$-linear problem for a circle by reducting it to a two-dimensional $\mathbb{C}$-linear problem.

2. Multiply Connected Domains

Theorem 1 (Bojarski [4])**.** *Let $|b(t)| < |a(t)|$. If $\kappa = \operatorname{wind}_{\partial D} a(t) \geq 0$, then problem* (1) *is solvable and the homogeneous problem* (1) ($c(t) = 0$) *has 2κ $\mathbb{R}$-linearly independent solutions vanishing at* *infinity. If $\kappa < 0$, then problem (1) has a unique solution if and only if $|2\kappa|$ $\mathbb{R}$-linearly independent conditions on $c(t)$ are fulfilled.*

Proof. This theorem was proved for simply connected domain in [4]. Now we prove it for multiply connected domains.

First, represent the coefficient $a(t)$ in the form [8]

$$a(t) = t^{\kappa} \frac{\chi^{+}(t)}{\chi^{-}(t)}, \tag{12}$$

where the function $\chi(z)$ is analytic in the domains D^{+} and D and Hölder-continuous in the closures of the considered domains, $\chi(z) \neq 0$ for all z. Then (1) becomes

$$\phi^{+}(t) = t^{\kappa}\phi^{-}(t) - \rho(t)\overline{\phi^{-}(t)} + c_1(t), \quad t \in L, \tag{13}$$

where

$$\phi(z) = \frac{\varphi(z)}{\chi(z)}, \quad z \in D^{+}, \quad z \in D,$$

$$c_1(t) = \frac{c(t)}{\chi^{+}(t)}, \quad \rho(t) = -\frac{b(t)\overline{\chi^{-}(t)}}{\chi^{+}(t)}, \quad t \in L.$$

Let $\kappa \geq 0$. Introduce the function

$$\psi^{-}(z) = z^{\kappa}\phi^{-}(z) - P_{\kappa-1}(z), \tag{14}$$

where

$$P_{\kappa-1}(z) = \sum_{k=0}^{n} P_k z^k$$

is a polynomial such that $\psi^{-}(z)$ vanishes at infinity. Then (13) takes the form

$$\phi^{+}(t) = \psi^{-}(t) - \rho(t)\overline{\psi^{-}(t)} + c_2(t), \quad t \in L, \tag{15}$$

where the function

$$c_2(t) = c_1(t) + P_{\kappa-1}(t) + t^{-\kappa}\rho(t)\overline{P_{\kappa-1}(t)}$$

$\mathbb{R}$-linearly depends on κ complex constants $P_0, P_1, \ldots, P_{\kappa-1}$, i.e., linearly depends on 2κ real constants $\operatorname{Re} P_0, \operatorname{Re} P_1, \ldots, \operatorname{Re} P_{\kappa-1}$ and $\operatorname{Im} P_0, \operatorname{Im} P_1, \ldots, \operatorname{Im} P_{\kappa-1}$. The coefficient $\rho(t)$ satisfies the inequality

$$|\rho(t)| = \left| \frac{b(t)\overline{\chi^{-}(t)}}{\chi^{+}(t)} \right| = \left| \frac{b(t)}{a(t)} \right| < 1, \quad t \in L.$$

This implies that the Noetherian $\mathbb{R}$-linear problem (15) has zero winding number. Hence, $\ell - p = 0$, where ℓ is the number of $\mathbb{R}$-linear independent solutions of (15) and p is the number of $\mathbb{R}$-linear independent solvability conditions on $c_2(t)$. Consider the $\mathbb{C}$-linear problem

$$\phi^{+}(t) = A(t)\psi^{-}(t), \quad t \in L, \tag{16}$$

where

$$A(t) = 1 - \rho(t)\frac{\overline{\psi^{-}(t)}}{\psi^{-}(t)}.$$

Problem (16) has only zero solution vanishing at infinity, since $\operatorname{wind}_L A(t) = 0$. Therefore, $\ell = p = 0$ and the inhomogeneous problem (15) has a unique solution for each fixed $c_2(t)$. Then 2κ arbitrary real constants $\operatorname{Re} P_0, \operatorname{Re} P_1, \ldots, \operatorname{Re} P_{\kappa-1}$ and $\operatorname{Im} P_0, \operatorname{Im} P_1, \ldots, \operatorname{Im} P_{\kappa-1}$ in $c_2(t)$ produce 2κ real linear independent solutions of (13).

Now let $\kappa < 0$. Introduce the function

$$\psi(z) = z^{\kappa}\phi(z) \tag{17}$$

with zero at least of order $-(\kappa+1)$ at infinity. Then (13) takes the form

$$\phi^+(t) = \psi^-(t) - \rho(t)\overline{\psi^-(t)} + c_1(t), \quad t \in L. \tag{18}$$

As follows from the above, this problem has a unique solution. Having solved it, we must impose the condition that $\psi(z)$ has zero at least of order $-(\kappa+1)$ at infinity. This condition yields 2κ real linear independent conditions on $c_1(t)$. The theorem is proved. □

Denote by $\mathcal{H}(\Gamma)$ the class of Hölder continuous functions on a smooth curve Γ.

Applying Theorem 1 to (6), we obtain the following result.

Corollary 2. *Let λ_k and λ be positive. Then problem (6) has a unique solution for any function $f \in \mathcal{H}(L)$.*

3. Integral Equations

There are two different methods of solving integral equations associated with boundary-value problems. The first method is known as the method of potentials. In complex analysis, it is equivalent to the method of singular integral equations [8, 29, 30, 34]. The other method of Schwarz can be presented as a method of integral equations of another type [19, 20]. In the present section, problem (6) is written in such a form.

Introduce the space $\mathcal{H}(D^+)$ consisting of functions that are analytic in $D^+ = \bigcup\limits_{k=1}^{n} D_k$ and Hölder-continuous in the closure of D^+ endowed with the norm

$$\|\omega\| = \inf_{t\in L} |\omega(t)| + \inf_{t_{1,2}\in L} \frac{|\omega(t_1)| - \omega(t_2)|}{|t_1 - t_2|^\alpha}, \tag{19}$$

where $0 < \alpha \leq 1$. The space $\mathcal{H}(D^+)$ is a Banach space since the norm in $\mathcal{H}(D^+)$ coincides with the norm of functions that are Hölder-continuous on L (inf on $D^+ \cup L$ in (19) is equal to inf on L). The Harnack theorem implies that convergence in the space $\mathcal{H}(D^+)$ implies uniform convergence in the closure of D^+.

For fixed m, introduce the operator

$$A_m f(z) = \frac{1}{2\pi i} \int\limits_{L_m} \frac{f(t)dt}{t-z}, \quad z \in D_m. \tag{20}$$

By the Sokhotsky formulas,

$$A_m f(\zeta) = \lim_{z\to\zeta} A_m f(z) = \frac{1}{2} f(\zeta) + \frac{1}{2\pi i} \int\limits_{L_m} \frac{f(t)dt}{t-\zeta}, \quad \zeta \in L_m. \tag{21}$$

Equations (20)–(21) determine the operator A_m in the space $\mathcal{H}(D_m)$.

Lemma 3. *The linear operator A_m is bounded in the space $\mathcal{H}(D_m)$.*

The proof is based on the definition of the bounded operator, $\|A_m f\| \leq C\|f\|$, and the fact that the norm in $\mathcal{H}(D_m)$ is equal to the norm of functions Hölder-continuous on L_m. The estimate of the latter norm follows from the boundness of operator (21) in the Hölder's space [8].

The conjugation condition (15) can be written in the form

$$\phi_k(t) - \phi^-(t) = \rho(t)\overline{\phi_k(t)} + c_2(t), \quad t \in L_k, \quad k = 1, 2, \dots, n, \tag{22}$$

where $\phi_k(z) = \psi(z)$ in $D_k \cup L_k$. The difference of functions analytic in D^+ and in D is in the left-hand siade of the last relation. Applying the Sokhotsky formulas, we have

$$\phi_k(z) = \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{L_m} \frac{\rho(t)\overline{\phi_m(t)}}{t-z} dt + f_k(z), \quad z \in D_k, \quad k = 1, 2, \ldots, n, \tag{23}$$

where the function

$$f_k(z) = \frac{1}{2\pi i} \sum_{m=1}^{n} \int_{L_m} \frac{c_2(t)}{t-z} \, dt$$

is analytic in D_k and Hölder-continuous in its closure. The integral equations (23) can be continued to L_k as follows:

$$\phi_k(z) = \sum_{m=1}^{n} \left[\frac{\rho(z)\overline{\phi_k(z)}}{2} + \frac{1}{2\pi i} \int_{L_m} \frac{\rho(t)\overline{\phi_m(t)}}{t-z} dt \right] + f_k(z), \quad z \in L_k, \quad k = 1, 2, \ldots, n. \tag{24}$$

One can consider Eqs. (23) and (24) as an equation with linear bounded operator in the space $\mathcal{H}(D^+)$.

Equations (23) and (24) correspond to the generalized method of Schwarz. Write, for instance, Eq. (23) in the form

$$\phi_k(z) - \frac{1}{2\pi i} \int_{L_k} \frac{\rho(t)\overline{\phi_k(t)}}{t-z} dt = \sum_{m \neq k} \frac{1}{2\pi i} \int_{L_m} \frac{\rho(t)\overline{\varphi_m(t)}}{t-z} dt + f_k(z), \quad z \in D_k, \quad k = 1, 2, \ldots, n. \tag{25}$$

At the zeroth approximation, we arrive at the problem for the single inclusion D_k, $k = 1, 2, \ldots, n$:

$$\phi_k(z) - \frac{1}{2\pi i} \int_{L_k} \frac{\rho(t)\overline{\phi_k(t)}}{t-z} dt = f_k(z), \quad z \in D_k. \tag{26}$$

Let problem (26) be solved. Further, its solution is substituted into the right-hand side of (25). Then we arrive at the first-order problem, etc. Therefore, the generalized method of Schwarz can be considered as a method of implicit iterations applied to the integral equations (23) and (24).

The method of integral equations was proposed in [27, Chap. 4] for the Dirichlet problem. The converging method of direct iterations for these equations coincides with the modified method of Schwarz. However, the integral terms of this method contain Green's functions of the domains D_k which should be constructed. One can obtain similar equations applying the operator $\mathcal{S}_k^{-1}$ to both sides of (25), where the operator $\mathcal{S}_k$ solves Eq. (26).

4. Method of Successive Approximations

We use the following general result from [15].

Theorem 4. *Let A be a linear bounded operator in a Banach space $\mathcal{B}$. If for any element $f \in \mathcal{B}$ and for any complex number ν satisfying the inequality $|\nu| \leq 1$, the equation*

$$x = \nu A x + f \tag{27}$$

has a unique solution, then a unique solution of the equation

$$x = Ax + f \tag{28}$$

can be found by the method of successive approximations. The approximations converge in $\mathcal{B}$ to the solution

$$x = \sum_{k=0}^{\infty} A^k f. \tag{29}$$

Theorem 4 can be applied to Eqs. (23) and (24).

Theorem 5. *Let $|\rho_k| < 1$. Then the system of Eqs.* (23) *and* (24) *has a unique solution. This solution can be found by the method of successive approximations convergent in the space $\mathcal{H}(D^+)$.*

Proof. Let $|\nu| \leq 1$. In $\mathcal{H}(D^+)$, consider the equations

$$\phi_k(z) = \nu \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{L_m} \frac{\rho(t)\overline{\phi_m(t)}}{t-z} dt + f_k(z), \quad z \in D_k, \quad k = 1, 2, \ldots, n. \tag{30}$$

Equations on L_k have a form similar to (24).

Let $\phi_k(z)$ be a solution of (30). Introduce the function $\phi(z)$ analytic in D and Hölder-continuous in its closure as follows:

$$\phi(z) = \nu \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{L_m} \frac{\rho(t)\overline{\phi_m(t)}}{t-z} dt, \quad z \in D. \tag{31}$$

The expression in the right-hand side of (31) can be considered as Cauchy's integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{L} \frac{\mu(t)}{t-z} dt$$

along $L = \bigcup_{m=1}^{n} L_m$ with the density $\mu(t) = \rho(t)\overline{\phi_k(t)}$ on L_k. Using the property of Cauchy's integral

$$\Phi^+(t) - \Phi^-(t) = \mu(t)$$

on L and (30), we arrive at the $\mathbb{R}$-linear conjugation relation on each fixed curve L_k:

$$\phi_k(t) - f_k(t) - \phi(t) = \rho(t)\overline{\phi_k(t)}, \quad t \in L_k. \tag{32}$$

Here

$$\Phi^+(t) = \phi_k(t) - f_k(t), \quad \Phi^-(t) = \phi(t).$$

In accordance with the Corollary 2 of Theorem 1, the $\mathbb{R}$-linear problem (31) has a unique solution. This unique solution is the unique solution of the system (30).

Theorem 4 yields the convergence of the method of successive approximations applied to system (30). The theorem is proved. □

5. Conclusion

Although the method of integral equations discussed in Sec. 4 is rather a numerical method, application of the residues for special forms of inclusions transforms integral terms into compositions of functions. Therefore, at least for boundaries expressed by algebraic functions, one should arrive at functional equations. An example concerning elliptical inclusions is presented in [25]. This approach can be considered as a generalization of Grave's method reviewed in [2] to multiply connected domains.

In order to understand the place of the convergence results obtained in this paper, we return to Sec. 1.2. It was established in the previous works that for $|b(t)| < |a(t)|$, the problem has a unique solution. If the stronger condition (9) is fulfilled (always $S_p \geq 1$), this unique solution can be constructed by the absolutely converging method of successive approximations. The absolute convergence implies geometrical restrictions on geometry which can be roughly presented as follows. Each inclusion D_k is sufficiently far away from other inclusions D_m ($m \neq k$). Only after the results presented in the present paper does the situation become clear and simplify. In the case (8), the method of successive approximations can be also applied, but absolute convergence is replaced by uniform convergence. The same situation with convergence repeats for other methods and problems. In all previous works, beginning from Poincaré's investigations, the Schwarz operator, the Poincaré series, the Riemann–Hilbert problem, the modified alternating Schwarz method, the Schwarz–Christoffel map [5], etc., were studied by absolutely convergent methods under geometrical restrictions. The main result of the present paper is based on the modification of these methods and studying the problems by uniformly convergent methods. This replacement of the absolute convergence by the uniform convergence abandons all previous geometrical restrictions and yields a solution to the problems and convergence of the methods for an arbitrary location of the nonoverlapping inclusions.

For engineers, it is interesting to get exact and approximate formulas for the effective conductivity tensor. One can find a description of such formulas based on the solution to the problems discussed in the present paper in the review [26].

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