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ℝ-linear and Riemann–Hilbert Problems for Multiply Connected Domains

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Abstract. The ℝ-linear problem with constant coefficients for arbitrary multiply connected domains has been solved. The method is based on reduction of the problem to a system of functional equations for a circular domain and to integral equations for a general domain. In previous works, the ℝ-linear problem and its partial cases such as the Riemann–Hilbert problem and the Dirichlet problem were solved under geometrical restrictions to the domains. In the present work, the solution is constructed for any circular multiply connected domain in the form of modified Poincaré series. Moreover, the modified alternating Schwarz method has been justified for an arbitrary multiply connected domain. This extends application of the alternating Schwarz method, since in the previous works geometrical restrictions were imposed on locations of the inclusions. The same concerns Grave's method which was worked out before only for simple closed algebraic boundaries or for a collection of confocal boundaries.

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1. Introduction

Various boundary value problems are reduced to singular integral equations [Gakhov (1977)], [Muskhelishvili (1968)], [Vekua (1988)]. Only some of them can be solved in closed form. In the present paper we follow the lines of the book [Mityushev and Rogosin (2000)] and describe application of the functional equations method to the ℝ-linear problem which in a particular case yields the Riemann–Hilbert problem.

This problem can be considered as a generalization of the classical Dirichlet and Neumann problems for harmonic functions. It includes as a particular case the mixed boundary value problem. We know the famous Poisson formula which solves the Dirichlet problem for a disk. The exact solution of the Dirichlet problem for a circular annulus is also known due to Villat–Dini (see [Koppenfels and Stallman (1959)], p. 119 and [Akhiezer (1990)], p. 169). Formulae from Theorems 3.6 and 3.7 presented below can be considered as a generalization of the Poisson and Villat–Dini formulae to arbitrary circular multiply connected domains. In order to deduce our formulae we first reduce the boundary value problem to the $\mathbb{R}$-linear problem and solve the later one by use of functional equations. By *functional equations* we mean iterative functional equations [Kuczma et al. (1990)], [Mityushev and Rogosin (2000)] with shift into its domain. Hence, we do not use traditional integral equations and infinite systems of linear algebraic equations. The solution is given explicitly in terms of the known functions or constants and geometric parameters of the domain.

Despite the solution being given by exact formulae, its structure is not elementary. More precisely, it is represented in the form of integrals involving the Abelian functions [Baker (1996)] (Poincaré series [Mityushev (1998)] or their counterparts [Mityushev and Rogosin (2000)]). The reason why the solution in general is not presented by integrals involving elementary kernels is of a topological nature. In order to explain this, we briefly recall the scheme of the solution to the Riemann–Hilbert problem

$$\phi(t) + G(t)\overline{\phi(t)} = g(t), \; t \in \partial\mathbb{C}^+, \tag{1.1}$$

for the upper half-plane $\mathbb{C}^+$ following [Gakhov (1977)], [Muskhelishvili (1968)]. Define the function $\phi^-(z) := \overline{\phi(\overline{z})}$ analytic in the lower half-plane. Then the Riemann–Hilbert problem (1.1) becomes the $\mathbb{C}$-linear problem (Riemann problem)

$$\phi^+(t) + G(t)\phi^-(t) = g(t), \; t \in \partial\mathbb{C}^+. \tag{1.2}$$

The latter problem is solved in terms of Cauchy type integrals (see details in [Gakhov (1977)], [Muskhelishvili (1968)]).

Let us look at this scheme from another point of view [Zverovich (1971)]. Introduce a copy of the upper half-plane $\mathbb{C}^+$ with the local complex coordinate $\overline{z}$ and glue it with $\mathbb{C}^+$ along the real axis. Define the function $\phi^-(\overline{z}) := \overline{\phi(z)}$ analytic on the copy of $\mathbb{C}^+$. Then we again arrive at the $\mathbb{C}$-linear problem (1.2) but on the double of $\mathbb{C}^+$ which is conformally equivalent to the Riemann sphere $\hat{\mathbb{C}}$. The fundamental functionals of $\hat{\mathbb{C}}$ are expressed by means of meromorphic functions which produces the Cauchy type integrals. The same scheme holds for any n-connected domain D. As a result, we arrive at problem (1.2) on the Schottky double of D, the Riemann surface of genus $(n-1)$, where life is more complicated than on the plane, i.e., on the Riemann sphere of zero genus. It is not described by meromorphic functions. Therefore, if one tries to solve problem (1.2) on the double of D, one has to use meromorphic analogies of the Cauchy kernel on Riemann surfaces, i.e., the Abelian functions. In the case $n = 2$, the double of D becomes a torus in which meromorphic functions are replaced by the classical elliptic functions [Akhiezer (1990)]. [Crowdy (2009)], [Crowdy (2008a)], [Crowdy (2008b)] used the

Schottky–Klein prime function associated with the Schottky double of D to solve many different problems for multiply connected domains.

The paper is organized as follows. First, we describe the known results, discuss the Riemann–Hilbert and the Schwarz problems. In Section 2 we discuss functional equations and prove convergence of the method of successive approximations for these equations. In Section 3 the harmonic measures of the circular multiply connected domains and the Schwarz operator are constructed by a method which can be outlined as follows. At the beginning the Schwarz problem is written as an $\mathbb{R}$-linear problem. Then we reduce it to functional equations. Application of the method of successive approximations yields the solution in the form of a Poincaré series of weight 2. As a sequence we obtain the almost uniform convergence of the Poincaré series for any multiply connected domain. The $\mathbb{R}$-linear problem with constant coefficients is studied in Section 4. Application of the method of successive approximations has been justified to the $\mathbb{R}$-linear problem for arbitrary multiply connected domain.

1.1. Riemann–Hilbert problem

Let D be a multiply connected domain on the complex plane whose boundary ∂D consists of n simple closed Jordan curves. The positive orientation on ∂D leaves D to the left. This orientation is kept up to Section 3.

The scalar linear Riemann–Hilbert problem for D is stated as follows: Given Hölder continuous functions $\lambda(t) \neq 0$ and $f(t)$ on ∂D, to find a function $\phi(z)$ analytic in D, continuous in the closure of D with the boundary condition

$$\mathrm{Re}\,\overline{\lambda(t)}\phi(t) = f(t),\ t \in \partial D. \tag{1.3}$$

This condition can also be written in the form (1.1).

The problem (1.3) had been completely solved for simply connected domains ($n = 1$). Its solution and general theory of boundary value problems is presented in the classic books [Gakhov (1977)], [Muskhelishvili (1968)] and [Vekua (1988)]. In 1975 [Bancuri (1975)] solved the Riemann–Hilbert problem for circular annulus ($n = 2$).

First results concerning the Riemann–Hilbert problem for general multiply connected domains were obtained [Kveselava (1945)]. He reduced the problem to an integral equation. Beginning in 1952, I.N. Vekua and later Bojarski began to extensively study this problem. Their results are presented in the book [Vekua (1988)]. This Georgian attack on the problem, supported by a young Polish mathematician, were successful. Due to Kveselava, Vekua and Bojarski, we have a theory of solvability of problem (1.3) based on integral equations and estimations of its defect numbers, l_χ, the number of linearly independent solutions and p_χ, the number of linearly independent conditions of solvability on $f(t)$. Here, $\chi = wind_{\partial D}\lambda$ is the index of the problem. In particular, Bojarski obtained the exact estimation $l_\chi \leq \chi + 1$. In the special case $0 < \chi < n - 2$, Bojarski showed that solvability of the problem depends on a system of linear algebraic equations with 2χ unknowns. It was also demonstrated that the rank of this system differs from 2χ

on the set of zeros of an analytic function of few variables. Hence, almost always $l_\chi = max(0, 2\chi - n + 2)$. In [Zverovich (1971)] the theory was developed by reduction of the problem (1.3) to the $\mathbb{C}$-linear problem (1.2) on the Riemann surface and it was shown that the solution of the problem is expressed in terms of the fundamental functionals of the double of D.

Any multiply connected domain D can be conformally mapped onto a circular multiply connected domain ([Golusin (1969)], p. 235). Hence, it is sufficient to solve the problem (1.3) for a circular domain and after to write the solvability conditions and solution using a conformal mapping. The complete solution to problem (1.3) for an arbitrary circular multiply connected domain had been given in [Mityushev (1994)], [Mityushev (1998)], [Mityushev and Rogosin (2000)] by the method of functional equations.

1.2. $\mathbb{R}$-linear problem

Let D be a multiply connected domain described above. Let D_k $(k = 1, 2, \ldots, n)$ be simply connected domains complementing D to the extended complex plane. In the theory of composites, the domains D_k are called by inclusions. The $\mathbb{R}$-*linear conjugation problem* or simply $\mathbb{R}$-*linear problem* is stated as follows. Given Hölder continuous functions $a(t) \neq 0$, $b(t)$ and $f(t)$ on ∂D. To find a function $\phi(z)$ analytic in $\cup_{k=1}^n D_k \cup D$, continuous in $D_k \cup \partial D_k$ and in $D \cup \partial D$ with the conjugation condition

$$\phi^+(t) = a(t)\phi^-(t) + b(t)\overline{\phi^-(t)} + f(t),\ t \in \partial D. \tag{1.4}$$

Here $\phi^+(t)$ is the limit value of $\phi(z)$ when $z \in D$ tends to $t \in \partial D$, $\phi^-(t)$ is the limit value of $\phi(z)$ when $z \in D_k$ tends to $t \in \partial D$. In the case $|a(t)| \equiv |b(t)|$ the $\mathbb{R}$-linear problem is reduced to the Riemann–Hilbert problem (1.3) [Mikhailov (1963)].

In the case of the smooth boundary ∂D, the homogeneous $\mathbb{R}$-linear problem with constant coefficients

$$\phi^+(t) = a\phi^-(t) + b\overline{\phi^-(t)},\ t \in \partial D \tag{1.5}$$

is equivalent to the *transmission problem* from the theory of harmonic functions

$$u^+(t) = u^-(t),\ \lambda^+ \frac{\partial u^+}{\partial n}(t) = \lambda^- \frac{\partial u^-}{\partial n}(t),\ t \in \partial D. \tag{1.6}$$

Here the real function $u(z)$ is harmonic in D and continuously differentiable in $D_k \cup \partial D_k$ and in $D \cup \partial D$, $\frac{\partial}{\partial n}$ is the normal derivative to ∂D. The conjugation conditions express the perfect contact between materials with different conductivities λ^+ and λ^-. The functions $\phi(z)$ and $u(z)$ are related by the equalities

$$\begin{aligned} u(z) &= \operatorname{Re}\phi(z),\ z \in D, \\ u(z) &= \frac{\lambda^- + \lambda^+}{2\lambda^+} \operatorname{Re}\phi(z),\ z \in D_k\ (k = 1, 2, \ldots, n). \end{aligned} \tag{1.7}$$

The coefficients are related by formulae (for details see [Mityushev and Rogosin (2000)], Sec. 2.12.)

$$a = 1,\ b = \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+}. \tag{1.8}$$

Let us note that for positive λ^+ and λ^- we arrive at the elliptic case $|b| < |a|$ in accordance with Mikhajlov's terminology [Mikhailov (1963)].

The non-homogeneous problem (1.4) with real coefficients $a(t)$ and $b(t)$ can be written as a transmission problem (1.6). If $a(t)$ and $b(t)$ are complex the transmission problem takes a more complicated form [Mikhailov (1963)].

In 1932, using the theory of potentials, [Muskhelishvili (1932)] (see also [Muskhelishvili (1966)], p. 522) reduced the problem (1.6) to a Fredholm integral equation and proved that it has a unique solution in the case $\lambda^\pm > 0$, the most interesting in applications. [Vekua and Rukhadze (1933)], [Vekua and Rukhadze (1933)] constructed a solution of (1.6) in closed form for an annulus and an ellipse (see also papers by Ruhadze quoted in [Muskhelishvili (1966)]). Hence, the paper [Muskhelishvili (1932)] is the first result on solvability of the $\mathbb{R}$-linear problem, [Vekua and Rukhadze (1933)] and [Vekua and Rukhadze (1933)] published in 1933 are the first papers devoted to exact solution of the $\mathbb{R}$-linear problem for an annulus and an ellipse. A little bit later [Golusin (1935)] considered the $\mathbb{R}$-linear problem in the form (1.6) by use of the functional equations for analytic functions (see below Section 1.4). Therefore, the paper [Golusin (1935)] is the first paper which concerns constructive solution to the $\mathbb{R}$-linear problem for special circular multiply connected domains. In further works these first results were not associated to the $\mathbb{R}$-linear problem even by their authors.

[Markushevich (1946)] had stated the $\mathbb{R}$-linear problem in the form (1.4) and studied it in the case $a(t) = 0$, $b(t) = 1$, $f(t) = 0$ when (1.4) is not a Nöther problem. Later [Muskhelishvili (1968)] (p. 455 in Russian edition) did not determine whether (1.4) was his problem (1.6) discussed in 1932 in terms of harmonic functions. [Vekua (1967)] established that the vector-matrix problem (1.4) is Nötherian if $\det a(t) \neq 0$.

[Bojarski (1960)] showed that in the case $|b(t)| < |a(t)|$ with $a(t)$, $b(t)$ belonging to the Hölder class $H^{1-\varepsilon}$ with sufficiently small ε, the $\mathbb{R}$-linear problem (1.4) is qualitatively similar to the $\mathbb{C}$-linear problem

$$\phi^+(t) = a(t)\phi^-(t) + f(t),\ t \in \partial D. \tag{1.9}$$

More precisely, Bojarski proved the following theorem for simply connected domains. His proof is also valid for multiply connected domains. Let $wind_L a(t)$ denote the winding number (index) of $a(t)$ along L:

Theorem 1.1 ([Bojarski (1960)]). *Let the coefficients of the problem* (1.4) *satisfy the inequality*

$$|b(t)| < |a(t)|. \tag{1.10}$$

If $\chi = wind_{\partial D} a(t) \geq 0$, *the problem* (1.4) *is solvable and the homogeneous problem* (1.4) ($f(t) = 0$) *has* 2χ $\mathbb{R}$*-linearly independent solutions vanishing at infinity. If*

$\chi < 0$, *the problem* (1.4) *has a unique solution if and only if* $|2\chi|$ $\mathbb{R}$*-linearly independent conditions on* $f(t)$ *are fulfilled.*

Later [Mikhailov (1963)] (first published in [Mikhailov (1961)]) developed this result to continuous coefficients $a(t)$ and $b(t)$; $f(t) \in \mathcal{L}^p(\partial D)$. The case $|b(t)| < |a(t)|$ was called the elliptic case. It corresponds to the partial case of the real constant coefficients a and b considered by [Muskhelishvili (1932)].

[Mikhailov (1963)] reduced the problem (1.4) to an integral equation and justified the absolute convergence of the method of successive approximation for the later equation in the space $\mathcal{L}^p(L)$ under the restrictions $wind_L a(t) = 0$ and

$$(1 + S_p)|b(t)| < 2|a(t)|, \tag{1.11}$$

where S_p is the norm of the singular integral in $\mathcal{L}^p(L)$. Further discussion of the conditions (1.10) and (1.11) is in our Conclusion.

1.3. Schwarz problem

As we noted above the Riemann–Hilbert problem (1.3) is a partial case of the $\mathbb{R}$-linear problem. Later we will need this fact in the case $a = 1$, $b = -1$.

Theorem 1.2. *The problem*

$$\operatorname{Re}\phi(t) = f(t), t \in \partial D \tag{1.12}$$

is equivalent to the problem

$$\phi^+(t) = \phi^-(t) - \overline{\phi^-(t)} + f(t),\ t \in \partial D, \tag{1.13}$$

i.e., the problem (1.12) *is solvable if and only if* (1.13) *is solvable. If* (1.12) *has a solution* $\phi(z)$, *it is a solution of* (1.13) *in* D *and a solution of* (1.13) *in* D_k *can be found from the following simple problem for the simply connected domain* D_k *with respect to function* $2\operatorname{Im}\phi^-(z)$ *harmonic in* D_k,

$$2\operatorname{Im}\phi^-(t) = \operatorname{Im}\phi^+(t) - f(t), t \in \partial D. \tag{1.14}$$

The problem (1.14) *has a unique solution up to an arbitrary additive real constant.*

The proof of the theorem is evident. We call problem (1.12) the *Schwarz problem* for the domain D. Along similar lines (1.14) is called the Schwarz problem for the domain D_k. The operator solving the Schwarz problem is called the *Schwarz operator* (in appropriate functional space). The function $v(z) = 2\operatorname{Im}\phi(z)$ is harmonic in D_k. Therefore, the Schwarz problem (1.14) is equivalent to the Dirichlet problem

$$v(t) = \operatorname{Im}\phi^+(t) - f(t),\ t \in \partial D.$$

For multiply connected domains D, the Schwarz problem (1.12) is not equivalent to a Dirichlet problem for harmonic functions, since any function harmonic in D is represented as the real part of a single-valued analytic function plus logarithmic terms (see for instance (3.2)).

The problem

$$\operatorname{Re}\phi(t) = f(t) + c_k,\ t \in \partial D_k,\ k = 1, 2, \ldots, n, \tag{1.15}$$

with undetermined constants c_k is called the *modified Schwarz problem*. The problem (1.15) always has a unique solution up to an arbitrary additive complex constant [Mikhlin (1964)].

1.4. Functional equations

[Golusin (1934)]–[Golusin (1935)] reduced the Dirichlet problem for circular multiply connected domains to a system of functional equations and applied the method of successive approximations to obtain its solution under some geometrical restrictions. Such a restriction can be roughly presented in the following form: each disk $\mathbb{D}_k$ lies sufficiently far away from all other disks $\mathbb{D}_m$ $(m \neq k)$. Golusin's approach was developed in [Zmorovich (1958)], [Dunduchenko (1966)], [Aleksandrov and Sorokin (1972)]. [Aleksandrov and Sorokin (1972)] extended Golusin's method to an arbitrary multiply connected circular domain. However, the analytic form of the Schwarz operator was lost. More precisely, the Schwarz problem was reduced via functional equations to an infinite system of linear algebraic equations. Application of the method of truncation to this infinite system was justified.[1]

We also reduce the problem to functional equations which are similar to Golusin's. The main advantage of our modified functional equations is based on the possibility to solve them without any geometrical restriction by successive approximations. It is worth noting that this solution produced the Poincaré series discussed above.

The same story repeats with the alternating Schwarz method, which we call for non-overlapping domains the *generalized Schwarz method* [Golusin (1934)], [Mikhlin (1964)]. It is also known as a *decomposition method* [Smith et al. (1996)]. [Mikhlin (1964)] developed the alternating Schwarz method to the Dirichlet problem for multiply connected domains and proved its convergence under some geometrical restrictions coinciding with Golusin's restrictions for circular domains. Having modified this method we obtained a method convergent for any multiply connected domain (for details see [Mityushev (1994)], [Mityushev and Rogosin (2000)]).

1.5. Poincaré series

Let us consider mutually disjointed disks $\mathbb{D}_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$ $(k = 1, 2, \ldots, n)$ in the complex plane $\mathbb{C}$. Let $\mathbb{D}$ be the complement of the closed disks $|z - a_k| \leq r_k$ to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e., $\mathbb{D} := \widehat{\mathbb{C}} \backslash \cup_{k=1}^{n} (\mathbb{D}_k \cup \partial\mathbb{D}_k)$. It is assumed that $\mathbb{T}_k \cap \mathbb{T}_m = \emptyset$ for $k \neq m$.

The circles $\mathbb{T}_k := \{t \in \mathbb{C} : |t - a_k| = r_k\}$ leave $\mathbb{D}$ to the left. Let

$$z_{(k)}^{*} = \frac{r_k^2}{\overline{z - a_k}} + a_k$$

[1] Though the method of truncation can be effective in numeric computations, one can hardly accept that this method yields a closed form solution. Any way it depends on the definition of the term "closed form solution". A regular infinite system [Kantorovich and Krylov (1958)] can be considered as an equation with compact operator, i.e., it is no more than a discrete form of a Fredholm integral equation.

be an inversion with respect to the circle $\mathbb{T}_k$. It is known that if $\Phi(z)$ is analytic in the disk $|z-a_k| < r_k$ and continuous in its closure, $\overline{\Phi(z^*_{(k)})}$ is analytic in $|z-a_k| > r_k$ and continuous in $|z-a_k| \geq r_k$.

Introduce the composition of successive inversions with respect to the circles $\mathbb{T}_{k_1}, \mathbb{T}_{k_2}, \dots, \mathbb{T}_{k_p}$,

$$z^*_{(k_p k_{p-1} \dots k_1)} := \left(z^*_{(k_{p-1} \dots k_1)} \right)^*_{(k_p)}. \tag{1.16}$$

In the sequence $k_1, k_2, \dots, k_p$ no two neighboring numbers are equal. The number p is called the *level* of the mapping. When p is even, these are Möbius transformations. If p is odd, we have anti-Möbius transformations, i.e., Möbius transformations in $\overline{z}$. Thus, these mappings can be written in the form

$$\begin{aligned} \gamma_j(z) &= (e_j z + b_j) / (c_j z + d_j), \ p \in 2\mathbb{Z}, \\ \gamma_j(\overline{z}) &= (e_j \overline{z} + b_j) / (c_j \overline{z} + d_j), \ p \in 2\mathbb{Z} + 1, \end{aligned} \tag{1.17}$$

where $e_j d_j - b_j c_j = 1$. Here $\gamma_0(z) := z$ (identical mapping with the level $p = 0$), $\gamma_1(\overline{z}) := z^*_{(1)}, \dots, \gamma_n(\overline{z}) := z^*_{(n)}$ (n simple inversions, $p = 1$), $\gamma_{n+1}(z) := z^*_{(12)}$, $\gamma_{n+2}(z) := z^*_{(13)}, \dots, \gamma_{n^2}(z) := z^*_{(n,n-1)}$ ($n^2 - n$ pairs of inversions, $p = 2$), $\gamma_{n^2+1}(\overline{z}) := z^*_{(121)}, \dots$ and so on. The set of the subscripts j of γ_j is ordered in such a way that the level p is increasing. The functions (1.17) generate a Schottky group $\mathcal{K}$. Thus, each element of $\mathcal{K}$ is presented in the form of a composition of inversions (1.16) or in the form of linearly ordered functions (1.17). Let $\mathcal{K}_m$ be such a subset of $\mathcal{K} \backslash \{\gamma_0\}$ that the last inversion of each element of $\mathcal{K}_m$ is different from $z^*_{(m)}$, i.e., $\mathcal{K}_m = \{z^*_{(k_p k_{p-1} \dots k_1)} : \ k_p \neq m\}$.

Let $H(z)$ be a rational function. This following series is called the Poincaré series:

$$\theta_{2q}(z) := \sum_{j=0}^{\infty} H(\gamma_j(z))(c_j z + d_j)^{-2q}, \ \ (q \in \mathbb{Z}/2) \tag{1.18}$$

for $q = 1$ associated with the subgroup $\mathcal{K}$.

Definition 1.3. A point z is called a limit point of the group $\mathcal{K}$ if z is a point of accumulation of the sequence $\gamma_j(z)$ for some $z \in \widehat{\mathbb{C}}$. A point which is not a limit point is called an ordinary point.

In other words, if z runs over the extended complex plane, then the accumulation points of the sequence $\gamma_j(z)$ generate the limit set $\Lambda(\mathcal{K})$. It is assumed that in the formula (1.18) $z \in B := \widehat{\mathbb{C}} \backslash (B_1 \cup \Lambda(\mathcal{K}))$, B_1 is the set of poles of all $H(\gamma_j(z))$ and $\gamma_j(z)$. Ordinary points are characterized by the following property.

Lemma 1.4. *A point z us a regular point of $\mathcal{K}$ if there exist numbers $k_1, k_2, \dots, k_m$ such that $z^*_{k_m k_{m-1} \dots k_1}$ belongs to $D \cup \partial D$.*

The points z_1 and z_2 are called congruent if there exists such $\gamma_j \in \mathcal{K}$ that $\gamma_j(z_1) = z_2$. All limit points of the Schottky group $\mathcal{K}$ lie within the disks $D \cup \partial D$. In the neighborhood of a limit point ς there is an infinite number of distinct points congruent to any point of $\widehat{\mathbb{C}}$ with, at most, exception of ς itself and of one other point. The limit set $\Lambda(\mathcal{K})$ is transformed into itself by any $\gamma_j \in \mathcal{K}$; $\Lambda(\mathcal{K})$ is closed and dense itself.

[Poincaré (1916)] introduced the θ_2-series (1.18) associated to various types of the Kleinian groups. He did not study carefully the Schottky groups and just conjectured that the corresponding θ_2-series always diverges [Poincaré (1916)], [Burnside (1891)] (p. 51). [Burnside (1891)] gave examples of convergent series for the Schottky groups (named by him the first class of groups) and studied their absolute convergence under some geometrical restrictions. In his study W. Burnside followed Poincaré's proof of the convergence of the θ_4-series. On p. 52 [Burnside (1891)] wrote "I have endeavoured to show that, in the case of the first class of groups, this series is convergent, but at present I have not obtained a general proof. I shall offer two partial proofs of the convergency; one of which applies only to the case of Fuchsian groups, and for that case in general, while the other will also apply to Kleinian groups, but only when certain relations of inequality are satisfied." Further, on p. 57 [Burnside (1891)] gave a condition for absolute convergence in terms of the coefficients of the Möbius transformations. He also noted that convergence holds if the radii of the circles $|z - a_k| = r_k$ are sufficiently less than the distances between the centers $|a_k - a_m|$ when $k \neq m$.

[Myrberg (1916)] gave examples of absolutely divergent θ_2-series. Afterwards many mathematicians justified the absolute convergence of the Poincaré series under geometrical restrictions to the locations of the circles (see for references [Crowdy (2008b)] and [Mityushev and Rogosin (2000)]). Here, we present such a typical restriction expressed in terms of the separated parameter Δ introduced by Henrici,

$$\Delta = \max_{k \neq m} \frac{r_k + r_m}{|a_k - a_m|} < \frac{1}{(n-1)^{\frac{1}{4}}} \tag{1.19}$$

for an n-connected domain $\mathbb{D}$ bounded by the circles $|z - a_k| = r_k$ $(k = 1, 2, \ldots, n)$.

Necessary and sufficient conditions for absolute and uniform convergence of the series have been found in [Akaza (1966)], [Akaza and Inoue (1984)] in terms of the Hausdorff dimension of $\Lambda(\mathcal{K})$. This result is based on the study of the series $\sum_{j=1}^{\infty} |c_j|^{-2}$.

After [Myrberg (1916)] it seemed that the opposite conjectures of Poincaré and Burnside were both wrong. However, it was proved in [Mityushev (1998)] that θ_2-series converges uniformly for any multiply connected domain $\mathbb{D}$ without any geometrical restriction that corresponds to Burnside's conjecture. The uniform convergence does not directly imply the automorphy relation, i.e., invariance under the Schottky group of transformations, since it is forbidden to change the order of summation without absolute convergence. But this difficulty can be easily overcome by using functional equations. As a result, the Poincaré series satisfies the required automorphy relation and can be written in each fundamental domain with a prescribed summation depending on this domain [Mityushev

(1998)]. The study [Mityushev (1998)] is based on the solution to a Riemann–Hilbert problem. First, the Riemann–Hilbert problem is written as an $\mathbb{R}$-linear problem which is stated as a conjugation problem between functions analytic in the disks $\mathbb{D}_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$ $(k = 1, 2, \ldots, n)$ and in $\mathbb{D}$. Further, the latter problem is reduced to a system of functional equations (without integral terms) with respect to the functions analytic in $|z - a_k| < r_k$. The method of successive approximations is justified for this system in a functional space in which convergence is uniform. Straightforward calculations of the successive approximations yields a Poincaré type series (see for instance (3.40) in this paper).

2. Linear functional equations

2.1. Homogeneous equation

Let G be a domain on the extended complex plane whose boundary ∂G consists of simple closed Jordan curves. Introduce the Banach space $\mathcal{C}(\partial G)$ of functions continuous on the curves of ∂G with the norm $||f|| = \max_{1 \le k \le n} \max_{\partial G} |f(t)|$. Let us consider a closed subspace $\mathcal{C}_{\mathcal{A}}(G)$ of $\mathcal{C}(\partial G)$ consisting of the functions analytically continued into all disks G. Further, we usually take $\cup_{k=0}^{n} \mathbb{D}_k$ and sometimes $\mathbb{D}$ as the domain G (not necessarily connected). For brevity, the notation $\mathcal{C}_{\mathcal{A}}$ for $\mathcal{C}_{\mathcal{A}}\left(\cup_{k=0}^{n} \mathbb{D}_k\right)$ is used.

Hereafter, a point $w \in \mathbb{D} \setminus \{\infty\}$ is fixed.

Lemma 2.1. *Let given numbers ν_k have the form $\nu_k := \exp(-i\mu_k)$ with $\mu_k \in \mathbb{R}$. Consider the system of functional equations with respect to the functions $\phi_k(z)$ analytic in $\mathbb{D}_k$,*

$$\phi_k(z) = -\nu_k \sum_{m \ne k} \overline{\nu_m} \left[\overline{\phi_m\left(z_{(m)}^*\right)} - \overline{\phi_m\left(w_{(m)}^*\right)} \right], \quad |z - a_k| \le r_k \ (k = 1, 2, \ldots, n). \tag{2.1}$$

This system has only the trivial solution.

Proof. Let $\phi_m(z)$ $(m = 1, 2, \ldots, n)$ be a solution of (2.1). Then the right-hand part of (2.1) implies that the function $\phi_k(z)$ is analytic in $|z - a_k| \le r_k$ $(k = 1, 2, \ldots, n)$. Introduce the function

$$\psi(z) := -\sum_{m=1}^{n} \overline{\nu_m} \left[\overline{\phi_m\left(z_{(m)}^*\right)} - \overline{\phi_m\left(w_{(m)}^*\right)} \right],$$

analytic in the closure of $\mathbb{D}$. Then the functions ψ, ϕ_k satisfy the $\mathbb{R}$-linear boundary conditions

$$\nu_k \psi(t) = \phi_k(t) - \overline{\phi_k(t)} + \overline{\phi_k\left(w_{(k)}^*\right)}, \quad |t - a_k| = r_k, \ \ k = 1, \ldots, n.$$

One can write the later relations in the following form:

$$\operatorname{Re} \nu_k \psi(t) = c_k, \ |t - a_k| = r_k, \ \ k = 1, \ldots, n, \tag{2.2}$$

$$2 \operatorname{Im} \phi_k(t) = \operatorname{Im} \nu_k \psi(t) + d_k, \ |t - a_k| = r_k, \ \ k = 1, \ldots, n. \tag{2.3}$$

Here $\phi_k\left(w^*_{(k)}\right) = c_k + id_k$. One may consider equalities (2.2) as a boundary value problem with respect to the function $\psi(z)$ analytic in $\mathbb{D}$ and continuous in its closure, i.e., $\psi \in \mathcal{C}_A(\mathbb{D})$. The real constants c_k have to be determined. We prove that the problem (2.2) has only constant solutions: $\psi(z) \equiv c,\ c_k = \operatorname{Re} \nu_k c$. Denote by $\psi(\mathbb{D}) := \left\{\varsigma \in \widehat{\mathbb{C}} : z \in \mathbb{D},\ \varsigma = \psi(z)\right\}$ the image of $\mathbb{D}$ under mapping ψ. It follows from the Boundary Correspondence Principle for conformal mapping that the boundary of $\psi(\mathbb{D})$ consists of the segments $\operatorname{Re} \nu_k \varsigma = c_k$ $(k = 1, 2, \ldots, n)$. But in this case the point $\varsigma = \infty \in \psi(\mathbb{D})$ corresponds to a point of $\mathbb{D}$. It contradicts boundedness of the function $\psi(z)$ in the closure of $\mathbb{D}$. Hence, $\phi(z) =$ constant and equalities (2.3) imply that $\phi_k(t) =$ constant [Gakhov (1977)]. Using (2.1) we have $\phi_k(z) \equiv 0$.

The lemma is proved. □

2.2. Non-homogeneous equation

Lemma 2.2. *Let $h \in \mathcal{C}_{\mathcal{A}},\ |\nu_k| = 1$. Then the system of functional equations*

$$\begin{gathered} \phi_k(z) = -\nu_k \sum_{m \neq k} \overline{\nu_m} \left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] + h_k(z), \\ |z - a_k| \leq r_k \ (k = 1, 2, \ldots, n), \end{gathered} \tag{2.4}$$

has a unique solution $\Phi \in \mathcal{C}_{\mathcal{A}}$. Here $\Phi(z) := \phi_k(z)$ in $|z - a_k| \leq r_k$, $k = 1, 2, \ldots, n$. This solution can be found by the method of successive approximations. The approximations converge in $\mathcal{C}_{\mathcal{A}}$.

Proof. Rewrite the system (2.4) on $\mathbb{T}_k$ in the form of a system of integral equations

$$\begin{gathered} \phi_k(t) = -\nu_k \sum_{m \neq k} \overline{\nu_m \frac{1}{2\pi i} \int_{\mathbb{T}_m^-} \phi_m(\tau) \left(\frac{1}{\tau - t^*_{(m)}} - \frac{1}{\tau - w^*_{(m)}}\right) d\tau} + h_k(t), \\ |t - a_k| = r_k \ (k = 1, 2, \ldots, n). \end{gathered} \tag{2.5}$$

The orientation on $\mathbb{T}_m^-$ leaves $\mathbb{D}_m$ to the left. The system (2.5) can be written as an equation in the space $\mathcal{C}\left(\cup_{k=1}^n \mathbb{T}_k\right)$:

$$\Phi = \mathbf{A}\Phi + h. \tag{2.6}$$

The integral operators from (2.5) are compact in $\mathcal{C}(\mathbb{T}_k)$; multiplication by $\overline{\nu_m}$ and complex conjugation are bounded operators in $\mathcal{C}$. Then $\mathbf{A}$ is a compact operator in $\mathcal{C}$. Since Φ is a solution of (2.6) in $\mathcal{C}$, hence $\Phi \in \mathcal{C}_{\mathcal{A}}$ (see Pumping principle from [Mityushev and Rogosin (2000)], Sec. 2.3). This follows from the properties of the Cauchy integral and the condition $h \in \mathcal{C}_{\mathcal{A}}$. Therefore, equation (2.6) in $\mathcal{C}$ and equation (2.4) in $\mathcal{C}_{\mathcal{A}}$ are equivalent when $h \in \mathcal{C}_{\mathcal{A}}$. It follows from Lemma 2.1 that the homogeneous equation $\Phi = \mathbf{A}\Phi$ has only a trivial solution. Then the Fredholm theorem implies that equation (2.6) or the system (2.4) has a unique solution.

Let us show the convergence of the method of successive approximations. By virtue of the Successive Approximation Theorem (see [Krasnosel'skii et al. (1969)] and [Mityushev and Rogosin (2000)], Sec. 2.3) it is sufficient to prove the inequality

$\rho(\mathbf{A}) < 1$, where $\rho(\mathbf{A})$ is the spectral radius of the operator $\mathbf{A}$. The inequality $\rho(\mathbf{A}) < 1$ is satisfied if for all complex numbers λ such that $|\lambda| \leq 1$, equation

$$\Phi = \lambda \mathbf{A} \Phi$$

has only a trivial solution. This equation can be rewritten in the form

$$\phi_k(z) = -\lambda \nu_k \sum_{m \neq k} \overline{\nu_m} \left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)} \right], \ |z - a_k| \leq r_k. \tag{2.7}$$

Consider the case $|\lambda| < 1$. Introduce the function, analytic in the closure of $\mathbb{D}$,

$$\psi(z) = -\lambda \sum_{m=0}^{n} \overline{\nu_m} \left(\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)} \right).$$

Then $\psi(z)$ and $\phi_k(z)$ satisfy the $\mathbb{R}$-linear problem

$$\nu_k \psi(t) = \phi_k(t) - \lambda \overline{\phi_k(t)} + \gamma_k, \ |t - a_k| = r_k, \ k = 1, 2, \ldots, n,$$

where $\gamma_k := \lambda \overline{\phi_k\left(w^*_{(k)}\right)}$. It can be written in the form

$$\nu_k \psi_0(t) = \phi_k(t) - \lambda \overline{\phi_k(t)} + \gamma_k - \nu_k \psi(\infty), \ |t - a_k| = r_k, \ k = 1, 2, \ldots, n, \tag{2.8}$$

where $\psi_0(z) = \psi(z) - \psi(\infty)$. Theorem 1.1 implies that problem (2.8) has the unique solution

$$\psi_0(z) = 0, \ \phi_k(z) = \frac{\gamma_k - \nu_k \psi(\infty) + \lambda(\overline{\gamma_k - \nu_k \psi(\infty)})}{|\lambda|^2 - 1}, \ k = 1, 2, \ldots, n.$$

Hence, $\phi_k(z) =$ constant. Then (2.7) yields $\phi_k(z) \equiv 0$.

Consider the case $|\lambda| = 1$. Then by substituting $\omega_k(z) = \phi_k(z)/\sqrt{\lambda}$ the system (2.7) is reduced to the same system with $\lambda = 1$. It follows from Lemma 2.1 that $\omega_k(z) = \phi_k(z) = 0$. Hence, $\rho(\mathbf{A}) < 1$.

This inequality proves the lemma. □

3. Schwarz operator

3.1. Harmonic measures

In the present section the number s is chosen from $1, 2, \ldots, n$ and fixed. The *harmonic measure* $\alpha_s(z)$ of the circle $\mathbb{T}_s$ with respect to $\partial\mathbb{D}$ is a function harmonic in $\mathbb{D}$, continuous in its closure, satisfying the boundary conditions

$$\alpha_s(t) = \delta_{sk}, \ |t - a_k| = r_k, \ k = 1, 2, \ldots, n, \tag{3.1}$$

where δ_{sk} is the Kronecker symbol. The functions α_s are infinitely $\mathbb{R}$-differentiable in the closure of $\mathbb{D}$ (see [Mityushev and Rogosin (2000)], Sec. 2.7.2). Using the Logarithmic Conjugation Theorem [Mityushev and Rogosin (2000)] we look for $\alpha_s(z)$ in the form

$$\alpha_s(z) = \operatorname{Re} \phi(z) + \sum_{m=1}^{n} A_m \ln|z - a_m| + A, \tag{3.2}$$

where A_m and A are real constants,

$$\sum_{m=1}^{n} A_m = 0. \tag{3.3}$$

The later condition follows from the limit in (3.2) as z tends to infinity. Using the boundary condition (3.1) and the representation (3.2) we arrive at the following boundary value problem:

$$\operatorname{Re}\phi(t) + \sum_{m=1}^{n} A_m \ln|t - a_k| + A = \delta_{sk},\ |t - a_k| = r_k. \tag{3.4}$$

This problem is equivalent to the ℝ-linear problem (see Introduction)

$$\phi(t) = \phi_k(t) - \overline{\phi_k(t)} + f_k(t),\ |t - a_k| = r_k,\ k = 1, 2, \ldots, n, \tag{3.5}$$

where the unknown functions $\phi \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}),\ \phi_k \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$,

$$\phi(w) = 0, \tag{3.6}$$

$$f_k(z) := \delta_{sk} - A - A_k \ln r_k - \sum_{m \neq k} A_m \ln(z - a_m), z \in \mathbb{D}_k. \tag{3.7}$$

The branch of $\ln(z - a_m)$ is fixed in such a way that the cut connecting the points $z = a_m$ and $z = \infty$ does not intersect the circles $\mathbb{T}_k$ for $k \neq m$ and does not pass through the point $z = w$. The function $f_k(z)$ satisfies the boundary condition

$$\operatorname{Re} f_k(t) := \delta_{sk} - A - \sum_{m=1}^{n} A_m \ln|t - a_m|,\ \ |t - a_k| = r_k$$

and belongs to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$.

Remark 3.1. More precisely the functions ϕ, ϕ_k and f_k are infinitely ℂ-differentiable in the closures of the domains considered.

Let us introduce the function

$$\Phi(z) := \begin{cases} \phi_k(z) + \sum_{m\neq k}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] - \overline{\phi_k\left(w^*_{(k)}\right)} + f_k(z), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |z - a_k| \le r_k, \\ \phi(z) + \sum_{m=1}^{n}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right],\ z \in \mathbb{D}. \end{cases}$$

Calculate the jump across the circle $\mathbb{T}_k$,

$$\Delta_k := \Phi^+(t) - \Phi^-(t),\ t \in \mathbb{T}_k,$$

where $\Phi^+(t) := \lim_{z\to t\ z\in\mathbb{D}} \Phi(z)$, $\Phi^-(t) := \lim_{z\to t\ z\in\mathbb{D}_k} \Phi(z)$. Using (3.5), (3.7) we get $\Delta_k = 0$. It follows from the Analytic Continuation Principle that $\Phi(z)$ is analytic in the extended complex plane. Then Liouville's theorem implies that $\Phi(z)$ is a constant. Using (3.6) we calculate $\Phi(w) = 0$, hence $\Phi(z) \equiv 0$. The definition of $\Phi(z) \equiv 0$ in $|z-a_k| \leq r_k$ yields the following system of functional equations:

$$\begin{aligned} \phi_k(z) = & -\sum_{m\neq k}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] - \delta_{sk} + A + A_k \ln r_k \\ & + \sum_{m\neq k} A_m \ln(z-a_m) + \overline{\phi_k\left(w^*_{(k)}\right)}, \quad |z-a_k| \leq r_k, \end{aligned} \tag{3.8}$$

with respect to the functions $\phi_k(z) \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$. The branches of logarithms are chosen in the same way as in (3.7).

The system of functional equations (3.8) is the main point to construct the harmonic measure α_s via the analytic function $\phi(z)$ by formula (3.2). If $\phi_k(z)$ are known, the required function $\phi(z)$ has the form

$$\phi(z) = -\sum_{m=1}^{n}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right], \; z \in \mathbb{D} \cup \partial\mathbb{D}. \tag{3.9}$$

It is convenient to represent $\phi_k(z)$ in the form

$$\phi_k(z) = \varphi_k^{(0)}(z) + \sum_{m=1}^{n} A_m \varphi_k^{(m)}(z), \tag{3.10}$$

where $\varphi_k^{(0)}(z)$ satisfies

$$\begin{aligned} \varphi_k^{(0)}(z) = & -\sum_{m\neq k}\left[\overline{\varphi_m^{(0)}\left(z^*_{(m)}\right)} - \overline{\varphi_m^{(0)}\left(w^*_{(m)}\right)}\right] - \delta_{sk} + A + A_k \ln r_k \\ & + \sum_{m\neq k} A_m \ln(w-a_m) + \overline{\phi_k\left(w^*_{(k)}\right)}, \quad |z-a_k| \leq r_k, \quad k=1,\ldots,n, \end{aligned} \tag{3.11}$$

$\varphi_k^{(m)}(z)$ satisfies

$$\begin{aligned} \varphi_k^{(m)}(z) = -\sum_{k_1\neq k}\left[\overline{\varphi_{k_1}^{(m)}\left(z^*_{(k_1)}\right)} - \overline{\varphi_{k_1}^{(m)}\left(w^*_{(k_1)}\right)}\right] + \delta'_{km} \ln \frac{z-a_m}{w-a_m}, \\ |z-a_k| \leq r_k, \quad k=1,\ldots,n \quad (m=1,\ldots,n). \end{aligned} \tag{3.12}$$

In (3.12), n systems of functional equations are written, m is the number of the system, $\delta'_{km} = 1-\delta_{km}$, where δ_{km} is the Kronecker symbol. It is assumed that the constants A, A_k and $\overline{\phi_k\left(w^*_{(k)}\right)}$ are fixed in (3.11). The values of these constants will be found later. According to Lemma 2.2, functional equations (3.11)–(3.12) can be solved by the method of successive approximations. The method of successive approximations applied to (3.12) yields

$$\varphi_k^{(m)}(z) = \delta'_{km} \ln \frac{z-a_m}{w-a_m} - \sum_{k_1 \neq k} \delta_{k_1}^{(m)} \ln \overline{\frac{\overline{z^*_{(k_1)}-a_m}}{w^*_{(k_1)}-a_m}} + \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \delta_{k_2}^{(m)} \ln \frac{z^*_{(k_2k_1)}-a_m}{w^*_{(k_2k_1)}-a_m} - \cdots, \tag{3.13}$$

where the sum $\sum\limits_{k_j \neq k_{j-1}}$ contains the terms with $k_j = 1,2,\ldots,n;\ k_j \neq k_{j-1}$. By virtue of Lemma 2.2 with $\nu_m = 1$ the series (3.13) converges uniformly in $|z-a_k| \leq r_k$. It follows from (3.11) that $\phi_k^{(0)}(z)$ $(k=1,2,\ldots,n)$ are constants, since the zeroth approximation is a constant and the operator from the right-hand side of (3.11) produces constants.

One can see from (3.9) that the constants $\varphi_k^{(0)}(z)$ do not impact on $\phi(z)$, hence using (3.10) we have

$$\phi(z) = -\sum_{m=1}^{n} A_m \sum_{k=1}^{n} \left[\overline{\varphi_k^{(m)}\left(z^*_{(k)}\right)} - \overline{\varphi_k^{(m)}\left(w^*_{(k)}\right)}\right]. \tag{3.14}$$

Substitution of (3.13) into (3.14) yields

$$\begin{aligned}\phi(z) = & -\sum_{m=1}^{n} A_m \sum_{k=1}^{n} \delta'_{km} \ln \overline{\frac{\overline{z^*_{(k)}-a_m}}{w^*_{(k)}-a_m}} + \sum_{m=1}^{n} A_m \sum_{k=1}^{n} \sum_{k_1 \neq k} \delta_{k_1}^{(m)} \ln \frac{z^*_{(k_1k)}-a_m}{w^*_{(k_1k)}-a_m} \\ & -\sum_{m=1}^{n} A_m \sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \delta_{k_2}^{(m)} \ln \overline{\frac{\overline{z^*_{(k_2k_1k)}-a_m}}{w^*_{(k_2k_1k)}-a_m}} + \cdots, \quad z \in \mathbb{D} \cup \partial\mathbb{D}.\end{aligned} \tag{3.15}$$

Using the properties of δ'_{km} one can rewrite (3.15) in the form

$$\begin{aligned}\phi(z) = & -\sum_{m=1}^{n} A_m \sum_{k \neq m} \ln \overline{\frac{\overline{z^*_{(k)}-a_m}}{w^*_{(k)}-a_m}} + \sum_{m=1}^{n} A_m \sum_{k=1}^{n} \sum_{k_1 \neq k, m} \ln \frac{z^*_{(k_1k)}-a_m}{w^*_{(k_1k)}-a_m} \\ & -\sum_{m=1}^{n} A_m \sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1, m} \ln \overline{\frac{\overline{z^*_{(k_2k_1k)}-a_m}}{w^*_{(k_2k_1k)}-a_m}} + \cdots, \quad z \in \mathbb{D} \cup \partial\mathbb{D}.\end{aligned} \tag{3.16}$$

In order to write (3.16) in more convenient form we use the following:

Lemma 3.2. *There holds the equality*

$$\sum_{k=1}^{n} \sum_{k_1 \neq k} \cdots \sum_{k_s \neq k_{s-1}, m} A_m R_{mk_sk_{s-1}\ldots k_1k} = \sum_{k_1 \neq m} \sum_{k_2 \neq k_1} \cdots \sum_{k_s \neq k_{s-1}} \sum_{k \neq k_s} A_m R_{mk_1k_2\ldots k_sk}. \tag{3.17}$$

Proof. It is sufficient to demonstrate that both parts of equality (3.17) contain the same terms. First, replace $k_1 k_2 \ldots k_s$ by $k_s k_{s-1} \ldots k_1$ in the right-hand part of (3.17) which becomes

$$\sum_{k_s \neq m} \sum_{k_{s-1} \neq k_s} \cdots \sum_{k_1 \neq k_2} \sum_{k \neq k_1} A_m R_{m k_s k_{s-1} \ldots k_1 k}. \tag{3.18}$$

The left-hand part of (3.17) can be written as the sum

$$\sum_{k=1}^{n} \sum_{k_1=1}^{n} \cdots \sum_{k_s=1}^{n} \delta'_{k_1 k} \delta'_{k_2 k_1} \cdots \delta'_{k_s k_{s-1}} \delta'_{k_s m} A_m R_{m k_s k_{s-1} \ldots k_1 k}, \tag{3.19}$$

where $\delta'_{lk} = 1 - \delta_{lk}$, δ_{lk} is the Kronecker symbol. One can see that the sum (3.18) written in the similar form (3.19) contains the same product of the complimentary Kronecker symbols.

This proves the lemma. □

Applying Lemma 3.2 to (3.16) we obtain

$$\begin{aligned} \phi(z) = & -\sum_{m=1}^{n} A_m \sum_{k \neq m} \ln \overline{\frac{\overline{z^*_{(k)} - a_m}}{w^*_{(k)} - a_m}} \\ & + \sum_{m=1}^{n} A_m \sum_{k_1 \neq m} \sum_{k \neq k_1} \ln \frac{z^*_{(k_1 k)} - a_m}{w^*_{(k_1 k)} - a_m} \\ & - \sum_{m=1}^{n} A_m \sum_{k_1 \neq m} \sum_{k_2 \neq k_1} \sum_{k \neq k_2} \ln \overline{\frac{\overline{z^*_{(k_1 k_2 k)} - a_m}}{w^*_{(k_1 k_2 k)} - a_m}} + \cdots, \quad z \in \mathbb{D} \cup \partial\mathbb{D}. \end{aligned} \tag{3.20}$$

It can be also written in the form

$$\phi(z) = \sum_{m=1}^{n} A_m \psi_m(z),$$

where

$$\begin{aligned} \psi_m(z) = & \ln \prod_{k \neq m} \overline{\frac{\overline{w^*_{(k)} - a_m}}{z^*_{(k)} - a_m}} + \ln \prod_{k_1 \neq m} \prod_{k \neq k_1} \frac{z^*_{(k_1 k)} - a_m}{w^*_{(k_1 k)} - a_m} \\ & + \ln \prod_{k_1 \neq m} \prod_{k_2 \neq k_1} \prod_{k \neq k_2} \overline{\frac{\overline{w^*_{(k_1 k_2 k)} - a_m}}{z^*_{(k_1 k_2 k)} - a_m}} + \cdots, \quad z \in \mathbb{D} \cup \partial\mathbb{D}. \end{aligned} \tag{3.21}$$

Let us rewrite (3.21) in terms of the group $\mathcal{K}$,

$$\psi_m(z) = \ln \left[\prod_{j \in \mathcal{K}_m}^{\infty} \psi_m^{(j)}(z) \right], \tag{3.22}$$

where

$$\psi_m^{(j)}(z) = \begin{cases} \frac{\gamma_j(z)-a_m}{\gamma_j(w)-a_m}, \text{ if level of } \gamma_j \text{ is even,} \\ \overline{\frac{\gamma_j(\overline{w})-a_m}{\gamma_j(\overline{z})-a_m}}, \text{ if level of } \gamma_j \text{ is odd.} \end{cases}$$

The numeration on j in (3.22) is fixed with increasing level.

In order to determine the constants A and A_m, substitute $z = w_{(k)}^*$ in the real parts of (3.8):

$$\begin{aligned} 0 = &- \sum_{m\neq k} \operatorname{Re}\left[\overline{\varphi_m\left(\left(w_{(k)}^*\right)_{(m)}^*\right)} - \overline{\varphi_m\left(w_{(m)}^*\right)}\right] \\ &- \delta_{sk} + A + A_k \ln r_k + \sum_{m\neq k} A_m \ln\left|w_{(k)}^* - a_m\right|, \ k = 1, 2, \ldots, n. \end{aligned} \tag{3.23}$$

The function φ_m has the form (3.13) and linearly depends on the unknown constants A_m. The equalities (3.3), (3.23) generate a system of $n+1$ linear algebraic equations with respect to $n+1$ unknowns $A, A_1, \ldots, A_n$. This system has a unique solution, since in the opposite case it contradicts the uniqueness of the solution to the Dirichlet problem.

Theorem 3.3. *The harmonic measures have the form*

$$\alpha_s(z) = \sum_{m=1}^{n} A_m \left[\operatorname{Re}\psi_m(z) + \ln|z - a_m|\right] + A, \tag{3.24}$$

where $\psi_m(z)$ is given in (3.22). The infinite product (3.22) converges uniformly on each compact subset of $\mathbb{D}\setminus\{\infty\}$. The real constants A and A_m are uniquely determined by the system (3.3), (3.23)).

Proof. Exact formulae for harmonic measures were deduced in a formal way. In order to justify them it is necessary to prove the change of the summation in (3.16) to obtain (3.20).

Using the designations of Section 2, we write the series (3.13) in the form

$$\Phi = \sum_{k=1}^{\infty} \mathbf{A}^k h, \tag{3.25}$$

where

$$\begin{aligned} h(z) = \delta'_{km} \ln\frac{z - a_m}{w - a_m}, \quad \mathbf{A}h(z) = -\sum_{k_1\neq k} \delta'_{k_1 m} \ln\overline{\frac{z_{(k_1)}^* - a_m}{w_{(k_1)}^* - a_m}}, \\ \mathbf{A}^2 h(z) = \sum_{k_1\neq k}\sum_{k_2\neq k_1} \delta'_{k_2 m} \ln\frac{z_{(k_2 k_1)}^* - a_m}{w_{(k_2 k_1)}^* - a_m} - \cdots. \end{aligned} \tag{3.26}$$

It is possible to change the order of summation in each term $\mathbf{A}^k h$ of the successive approximations which contains inversions only of the level k. Therefore, it is possible to change the order of summation in the series (3.15) in terms having the same level. The series (3.20) is obtained from (3.16) by application of this rule.

This proves the theorem. □

Remark 3.4. The constants A and A_m depend on the choice of w.

Remark 3.5. The logarithmic terms in (3.24) can be included into infinite product (3.21). Then (3.24) becomes

$$\alpha_s(z) = \sum_{m=1}^{n} A_m \ln \prod_{j\in\mathcal{K}_m\cup\{0\}} |\psi_m^{(j)}(z)| + A_0,$$

where $A_0 := A - \sum_{m=1}^{n} A_m \ln |w - a_m|$.

3.2. Exact formula for the Schwarz operator

Following the previous section we construct the complex Green function $M(z,\zeta)$ and the Schwarz operator for the circular multiply connected domain $\mathbb{D}$. One can find general properties of the Schwarz operator in [Mityushev and Rogosin (2000)], [Mikhlin (1964)].

Let z and ζ belong to the closure of $\mathbb{D}$. The real *Green function* $G(z,\zeta) = g(z,\zeta) - \ln|z-\zeta|$ is introduced via the function $g(z,\zeta)$ harmonic in $\mathbb{D}$ satisfying the Dirichlet problem

$$g(t,\zeta) - \ln|t-\zeta| = 0, \quad |t - a_k| = r_k \ (k = 1,2,\dots,n) \tag{3.27}$$

with respect to the first variable. If $G(z,\zeta)$ is known, the solution of the Dirichlet problem

$$u(t) = f(t), \ |t - a_k| = r_k \ (k = 1,2,\dots,n) \tag{3.28}$$

has the form

$$u(z) = \frac{1}{2\pi}\sum_{k=1}^{n}\int_{\mathbb{T}_k} f(\zeta)\frac{\partial G}{\partial \nu}(z,\zeta)d\sigma, \tag{3.29}$$

where ν is the outward (in sense of orientation) normal vector at the point $\zeta \in \partial\mathbb{D}$.

The *complex Green function* $M(z,\zeta)$ is defined by the formula

$$M(z,\zeta) = G(z,\zeta) + iH(z,\zeta), \tag{3.30}$$

where the function $H(z,\zeta)$ is harmonically conjugated to $G(z,\zeta)$ on the variable z. It has the form

$$H(z,\zeta) = \int_w^z -\frac{\partial G}{\partial y}dx + \frac{\partial G}{\partial x}dy$$

with $z = x + iy$.

Introduce the *Schwarz kernel* (see [Mityushev and Rogosin (2000)], Sec. 2.7.2)

$$T(z,\zeta) = \frac{\partial M}{\partial \nu}(z,\zeta), \quad \zeta \in \mathbb{D}. \tag{3.31}$$

In accordance with (3.28)–(3.31) the function

$$F(z) = \frac{1}{2\pi}\sum_{k=1}^{n}\int_{\mathbb{T}_k} f(\zeta)T(z,\zeta)d\sigma \tag{3.32}$$

satisfies the boundary value problem

$$\operatorname{Re} F(t) = f(t),\ |t - a_k| = r_k\ (k = 1, 2, \dots, n). \tag{3.33}$$

Here $u(z)$ from (3.29) is the real part of the analytic function $F(z)$ having in general multi-valued imaginary part in the multiply connected domain $\mathbb{D}$. If we are looking for single-valued $F(z)$ by (3.33), we arrive at the Schwarz problem in accordance with the terminology introduced in our Introduction.

We use the representation for the Green function (see [Mityushev and Rogosin (2000)], Sec. 2.7.2)

$$M(z, \zeta) = M_0(z, \zeta) + \sum_{k=1}^{n} \alpha_k(\zeta) \ln(z - a_k) - \ln(\zeta - z) + A(\zeta), \tag{3.34}$$

where α_k is a harmonic measure of $\mathbb{D}$, $A(\zeta)$ is a real function in ζ. The point w and the branches of $\ln(z - a_k)$ are fixed as in the previous section. Using (3.31), (3.34) we obtain

$$T(z, \zeta) = \frac{\partial M_0}{\partial \nu}(z, \zeta) + \sum_{m=1}^{n} \frac{\partial \alpha_m}{\partial \nu}(\zeta) \ln(z - a_m) - \frac{1}{\zeta - z}\frac{\partial \zeta}{\partial \nu} + \frac{\partial A}{\partial \nu}(\zeta). \tag{3.35}$$

The function $M_0(z, \zeta)$ is infinitely $\mathbb{C}$-differentiable in the closure of $\mathbb{D}$ in z and satisfies the boundary value problem which follows from (3.27) and (3.30),

$$\begin{gathered} \operatorname{Re}\left[M_0(t, \zeta) + \sum_{k=1}^{n} \alpha_k(\zeta) \ln(t - a_k) - \ln(\zeta - t) + A(\zeta)\right] = 0, \\ |t - a_k| = r_k,\ k = 1, 2, \dots, n;\ M_0(w, \zeta) = 0. \end{gathered} \tag{3.36}$$

The problem (3.36) has a unique solution. It is reduced to the following system of functional equations:

$$\begin{aligned} \phi_k(z) = &- \sum_{m \neq k} \left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] - \ln(\zeta - z) + A \\ &+ \alpha_k(\zeta) \ln r_k + \sum_{m \neq k} \alpha_m(\zeta) \ln(z - a_m) + \overline{\phi_k\left(w^*_{(k)}\right)}, \\ &|z - a_k| \le r_k,\ k = 1, \dots, n \end{aligned} \tag{3.37}$$

where $\phi_k(z)$ belongs to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ and $\mathbb{C}$-infinitely differentiable in the closure of $\mathbb{D}_k$. Here ζ is considered as a parameter fixed in the closure of $\mathbb{D}$. The required function $M_0(z, \zeta)$ is related to the auxiliary functions $\phi_k(z)$ by equality

$$M_0(z, \zeta) = -\sum_{k=1}^{n} \left[\overline{\phi_k\left(z^*_{(k)}\right)} - \overline{\phi_k\left(w^*_{(k)}\right)}\right],\ z \in \mathbb{D} \cup \partial\mathbb{D} \setminus \{\zeta\}. \tag{3.38}$$

In order to solve (3.37) we consider two auxiliary systems of functional equations

$$\Psi_k(z) = -\sum_{m\neq k}\left[\overline{\Psi_m\left(z_m^*\right)} - \overline{\Psi_m\left(w_m^*\right)}\right] + A + \alpha_k(\zeta)\ln r_k$$
$$+\sum_{m\neq k}\alpha_m(\zeta)\ln(z-a_m) + \overline{\phi_m\left(w_{(k)}^*\right)},$$
$$\Omega_k(z) = -\sum_{m\neq k}\left[\overline{\Omega_m\left(z_m^*\right)} - \overline{\Omega_m\left(w_m^*\right)}\right] - \ln(\zeta - z),\ |z-a_k|\le r_k,\ k=1,\dots,n.$$

The first system coincides with the system (2.5) ($\nu_k = 1$), and thus can be solved by the method of successive approximations (cf. Lemma 2.2). Let us consider the second system. If $|\zeta - z| \le r_s$ for some s, the right-hand part of the second system $-\ln(\zeta - z)$ does not belong to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_s)$. But by introducing a new unknown function $\Omega_s^{(0)}(z) := \Omega_s(z) - \ln(\zeta - z)$ we get an equation in the space $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_s)$. Therefore, the method of successive approximations can be applied to the second system too, and

$$\overline{\Psi_k\left(z_k^*\right)} - \overline{\Psi_k\left(w_k^*\right)} = \sum_{m\neq k}\alpha_m(\zeta)\ln\overline{\frac{z_{(k)}^* - a_m}{w_{(k)}^* - a_m}} - \sum_{k_1\neq k}\sum_{m\neq k_1}\alpha_m(\zeta)\ln\frac{z_{(k_1k)}^* - a_m}{w_{(k_1k)}^* - a_m}$$
$$+\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{m\neq k_2}\alpha_m(\zeta)\ln\overline{\frac{z_{(k_2k_1k)}^* - a_m}{w_{(k_2k_1k)}^* - a_m}} - \cdots, \tag{3.39}$$
$$\overline{\Omega_k\left(z_k^*\right)} - \overline{\Omega_k\left(w_k^*\right)} = \ln\overline{\frac{\zeta - w_{(k)}^*}{\zeta - z_{(k)}^*}} + \sum_{k_1\neq k}\ln\frac{\zeta - z_{(k_1k)}^*}{\zeta - w_{(k_1k)}^*} + \sum_{k_1\neq k}\sum_{k_2\neq k_1}\ln\overline{\frac{\zeta - z_{(k_2k_1k)}^*}{\zeta - w_{(k_2k_1k)}^*}}$$
$$+\cdots,\ z\in\mathbb{D}\cup\partial\mathbb{D}. \tag{3.40}$$

The series (3.39), (3.40) converge uniformly in every compact subset of $\mathbb{D}\cup\partial\mathbb{D}\setminus\{\zeta\}$. We have $\phi_k(z) = \Psi_k(z) + \Omega_k(z)$, hence the values

$$\overline{\phi_k\left(z_k^*\right)} - \overline{\phi_k\left(w_k^*\right)} = \overline{\Psi_k\left(z_k^*\right)} - \overline{\Psi_k\left(w_k^*\right)} + \overline{\Omega_k\left(z_k^*\right)} - \overline{\Omega_k\left(w_k^*\right)} \tag{3.41}$$

are completely determined. It follows from (3.38) that

$$M_0(z,\zeta) = \sum_{m=1}^{n}\alpha_m(\zeta)\psi_m(z) - \omega(z,\zeta),$$

where the functions $\psi_m(z)$ have the form (3.21) or (3.22), $\alpha_m(\zeta)$ are given in Theorem 3.3,

$$\omega(z,\zeta) = \ln\left(\prod_{k=1}^{n}\overline{\frac{\zeta - z_{(k)}^*}{\zeta - w_{(k)}^*}}\right)\left(\prod_{k=1}^{n}\prod_{k_1\neq k}\frac{\zeta - w_{(k_1k)}^*}{\zeta - z_{(k_1k)}^*}\right)\left(\prod_{k=1}^{n}\prod_{k_1\neq k}\prod_{k_2\neq k_1}\overline{\frac{\zeta - z_{(k_2k_1k)}^*}{\zeta - w_{(k_2k_1k)}^*}}\right)\cdots. \tag{3.42}$$

This infinite product can be represented in the form

$$\omega(z,\zeta) = \ln \prod_{j=1}^{\infty} \omega_j(z,\zeta), \tag{3.43}$$

where

$$\omega_j(z,\zeta) = \begin{cases} \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)}, & \text{if level of } \gamma_j \text{ is even,} \\ \frac{\overline{\zeta-\gamma_j(\overline{w})}}{\overline{\zeta-\gamma_j(\overline{z})}}, & \text{if level of } \gamma_j \text{ is odd.} \end{cases}$$

In order to find $A(\zeta)$ we substitute $w^*_{(k)}$ in the real part of (3.37) and obtain

$$\begin{aligned} 0 = & -\sum_{m\neq k} \operatorname{Re}\left[\overline{\phi_m\left(\left(w^*_{(k)}\right)^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] - \ln\left|\zeta - w^*_{(k)}\right| + A(\zeta) \\ & + \alpha_k(\zeta)\ln r_k + \sum_{m\neq k}\alpha_m(\zeta)\ln\left|w^*_{(k)} - a_m\right|,\ k = 1,\dots,n. \end{aligned} \tag{3.44}$$

The harmonic measures satisfy the equality

$$\sum_{m=1}^{n} \alpha_m(\zeta) = 1. \tag{3.45}$$

One can consider (3.44), (3.45) as a system of $n+1$ real linear algebraic equations with respect to $n+1$ real unknowns $\alpha_1(\zeta),\ \alpha_2(\zeta),\dots,\ \alpha_n(\zeta),\ A(\zeta)$. The systems (3.44), (3.45) and (3.3), (3.23) have the same homogeneous part. Therefore, the system (3.44), (3.45) has a unique solution We may at the beginning look for the complex Green function $M(z,\zeta)$ with undetermined periods $\alpha_k(\zeta)/2\pi$, find $\alpha_k(\zeta)$ from (3.44), (3.45) and after assert that $\alpha_k(\zeta)$ is a harmonic measure. In order to determine $A(\zeta)$, we fix for instance $k=n$ in (3.44) and obtain

$$\begin{aligned} A(\zeta) = & \sum_{m=1}^{n-1} \operatorname{Re}\left[\overline{\phi_m\left(\left(w^*_{(n)}\right)^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] + \ln\left|\zeta - w^*_{(n)}\right| \\ & - \alpha_k(\zeta)\ln r_k - \sum_{m=1}^{n-1}\alpha_m(\zeta)\ln\left|w^*_{(k)} - a_m\right|, \end{aligned} \tag{3.46}$$

where $\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right]$ has the form (3.39), (3.40), (3.41).

The function (3.32) is single-valued in $\mathbb{D}$ if and only if

$$\sum_{k=1}^{n} \int_{\mathbb{T}_k} f(\zeta)\frac{\partial\alpha_m}{\partial\nu}(\zeta)d\sigma = 0,\ m = 1,2,\dots,n. \tag{3.47}$$

Note that one of the relations (3.47) follows from the other ones. For instance, let (3.47) be valid for $m = 1,2,\dots,n-1$. Then (3.47) for $m=n$ is fulfilled, since

$$\sum_{k=1}^{n} \int_{\mathbb{T}_k} f(\zeta)\frac{\partial\alpha_n}{\partial\nu}(\zeta)d\sigma = -\sum_{m=1}^{n-1}\sum_{k=1}^{n} \int_{\mathbb{T}_k} f(\zeta)\frac{\partial\alpha_m}{\partial\nu}(\zeta)d\sigma = 0.$$

Here the identity (3.45) is used. With the help of (3.35) the single- and the multi-valued components of the Schwarz operator can be separated:

$$T(z,\zeta) = T_s(z,\zeta) + T_m(z,\zeta),$$
$$T_s(z,\zeta) = \sum_{m=1}^{n} \frac{\partial \alpha_m}{\partial \nu}(\zeta)\left[\psi_m(z) + \ln(z-a_m)\right],$$
$$T_m(z,\zeta) = \frac{\partial \omega}{\partial \nu}(z,\zeta) - \frac{1}{\zeta - z}\frac{\partial \zeta}{\partial \nu} + \frac{\partial A}{\partial \nu}(\zeta).$$

We now proceed to calculate the normal derivatives in the later formulae. One can see that

$$\frac{\partial f}{\partial \nu} d\sigma = -\frac{1}{i}\left[\frac{\partial f}{\partial \zeta} + \left(\frac{r_k}{\zeta - a_k}\right)^2 \frac{\partial f}{\partial \overline{\zeta}}\right] d\tau,\ |\zeta - a_k| = r_k, \tag{3.48}$$

for any $f \in \mathcal{C}^1(\partial \mathbb{D})$. Recall that we deal with the outward normal to $\mathbb{D}$. In order to apply (3.48) to $\omega(z,\zeta)$ we find from (3.42) that

$$\begin{aligned}\frac{\partial \omega}{\partial \zeta}(z,\zeta) &= \sum_{k=1}^{n}\sum_{k_1 \neq k}\left(\frac{1}{\zeta - w^*_{(k_1 k)}} - \frac{1}{\zeta - z^*_{(k_1 k)}}\right) \\ &\quad + \sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3 \neq k_2}\left(\frac{1}{\zeta - w^*_{(k_3 k_2 k_1 k)}} - \frac{1}{\zeta - z^*_{(k_3 k_2 k_1 k)}}\right) + \cdots \\ &= \sum_{j=1}^{\infty}{}'' \left(\frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)}\right),\end{aligned} \tag{3.49}$$

where the terms in the later sum are ordered due to increasing even level. We also have

$$\frac{\partial \omega}{\partial \overline{\zeta}}(z,\tau) = \sum_{j=1}^{\infty}{}' \left(\frac{1}{\overline{\zeta - \gamma_j(\overline{z})}} - \frac{1}{\overline{\zeta - \gamma_j(\overline{w})}}\right), \tag{3.50}$$

where elements γ_j have the odd level. Substituting (3.49), (3.50) into (3.35), (3.32) we arrive at the following

Theorem 3.6. *The Schwarz operator of* $\mathbb{D}$ *has the form*

$$\begin{aligned}\phi(z) &= \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{T}_k} f(\zeta)\left\{\sum_{j=2}^{\infty}{}''\left[\frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)}\right]\right. \\ &\quad \left. + \left(\frac{r_k}{\zeta - a_k}\right)^2 \sum_{j=1}^{\infty}{}' \left[\frac{1}{\overline{\zeta - \gamma_j(\overline{z})}} - \frac{1}{\overline{\zeta - \gamma_j(\overline{w})}}\right] - \frac{1}{\zeta - z}\right\} d\zeta \\ &\quad + \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{T}_k} f(\zeta)\frac{\partial A}{\partial \nu}(\zeta) d\sigma + \sum_{m=1}^{n} A_m\left[\ln(z-a_m) + \psi_m(z)\right] + i\varsigma,\end{aligned} \tag{3.51}$$

where

$$A_m := \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{T}_k} f(\zeta)\frac{\partial \alpha_m}{\partial \nu}(\zeta)d\sigma,\ m=1,2,\ldots,n,$$

$A(\zeta)$ has the form (3.46), The functions $\alpha_m(\zeta)$ and $\psi_m(z)$ are derived in Theorem 3.3, ς is an arbitrary real constant, $\sum'$ contains γ_j of odd level, $\sum''$ – of even level. The series converges uniformly in each compact subset of $\mathbb{D}\cup\partial\mathbb{D}\setminus\{\infty\}$.

The single-valued part of the Schwarz operator can be determined by solution of the modified Dirichlet problem (see [Mityushev and Rogosin (2000)], Sec. 2.7.2):

$$\operatorname{Re}\phi(t) = f(t) + c_k,\ t\in\mathbb{T}_k,\ k=1,2,\ldots,n, \tag{3.52}$$

where a given function $f\in\mathcal{C}(\partial\mathbb{D})$, c_k are undetermined real constants. If one of the constants c_k is fixed arbitrarily, the remaining ones are determined uniquely and $\phi(z)$ is determined up to an arbitrary additive purely imaginary constant (see [Mityushev and Rogosin (2000)], Sec. 2.7.2). Thus, we have

Theorem 3.7. *The single-valued part of the Schwarz operator of $\mathbb{D}$ corresponding to the modified Dirichlet problem (3.52) has the form*

$$\begin{aligned}\phi(z) = \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{T}_k}(f(\zeta)+c_k)&\left\{\sum_{j=2}^{\infty}{}''\left[\frac{1}{\zeta-\gamma_j(w)}-\frac{1}{\zeta-\gamma_j(z)}\right]\right.\\ &\left.+\left(\frac{r_k}{\zeta-a_k}\right)^2\sum_{j=1}^{\infty}{}'\left[\frac{1}{\overline{\zeta-\gamma_j(\overline{z})}}-\frac{1}{\overline{\zeta-\gamma_j(\overline{w})}}\right]-\frac{1}{\zeta-z}\right\}d\zeta\\ &+\frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{T}_k} f(\zeta)\frac{\partial A}{\partial\nu}(\zeta)d\sigma + i\varsigma.\end{aligned} \tag{3.53}$$

One of the real constants c_k can be fixed arbitrarily, the remaining ones are determined uniquely from the linear algebraic system

$$\sum_{k=1}^{n}\int_{\mathbb{T}_k}(f(\zeta)+c_k)\frac{\partial\alpha_m}{\partial\nu}(\zeta)d\sigma = 0,\ m=1,2,\ldots,n-1. \tag{3.54}$$

4. ℝ-linear problem

4.1. Integral equations

There are two different methods of integral equations associated to boundary value problems. The first method is known as the method of potentials. In complex analysis, it is equivalent to the method of singular integral equations [Gakhov (1977), Muskhelishvili (1968), Muskhelishvili (1966), Vekua (1988)]. The alternating method of Schwarz can be presented as a method of integral equations of another type [Mikhlin (1964), Mikhailov (1963)]. Let $L_k = \partial D_k$ be Lyapunov's simple closed curves. It is convenient to introduce the opposite orientation to the orientation considered in the above sections. So, it is assumed that each L_k leaves the inclusion D_k on the left. In the present section, we discuss the following $\mathbb{R}$-linear problem corresponding to the perfect contact between components of the composite with the external field $f(t)$,

$$\varphi^-(t) = \varphi_k(t) - \rho_k\overline{\varphi_k(t)} - f(t), \quad t \in L_k \ (k = 1, 2, \ldots, n). \tag{4.1}$$

Here, the contrast parameter $\rho_k = \frac{\lambda_k+\lambda}{\lambda_k-\lambda}$ is introduced via the conductivity of the host λ and the conductivity of the kth inclusion λ_k. Introduce a space $\mathcal{H}(D^+)$ consisting of functions analytic in $D^+ = \cup_{k=1}^n D_k$ and Hölder continuous in the closure of D^+ endowed with the norm

$$||\omega|| = \sup_{t\in L}|\omega(t)| + \sup_{t_{1,2}\in L}\frac{|\omega(t_1)| - \omega(t_2)|}{|t_1 - t_2|^\alpha}, \tag{4.2}$$

where $0 < \alpha \leq 1$. The space $\mathcal{H}(D^+)$ is Banach, since the norm in $\mathcal{H}(D^+)$ coincides to the norm of functions Hölder continuous on L (inf on $D^+ \cup L$ in (4.2) is equal to inf on L). It follows from Harnack's principle that convergence in the space $\mathcal{H}(D^+)$ implies uniform convergence in the closure of D^+.

For fixed m introduce the operator

$$A_m f(z) = \frac{1}{2\pi i}\int_{L_m}\frac{f(t)dt}{t-z}, \ z \in D_m. \tag{4.3}$$

In accordance with Sokhotskij's formulae,

$$A_m f(\zeta) = \lim_{z\to\zeta} A_m f(z) = \frac{1}{2}f(\zeta) + \frac{1}{2\pi i}\int_{L_m}\frac{f(t)dt}{t-\zeta}, \ \zeta \in L_m. \tag{4.4}$$

Equations (4.3)–(4.4) determine the operator A_m in the space $\mathcal{H}(D_m)$.

Lemma 4.1. *The linear operator A_m is bounded in the space $\mathcal{H}(D_m)$.*

The proof is based on a definition of the bounded operator $||A_m f|| \leq C||f||$ and the fact that the norm in $\mathcal{H}(D_m)$ is equal to the norm of functions Hölder continuous on L_m. The estimation of the later norm follows from the boundness of the operator (4.4) in Hölder's space [Gakhov (1977)].

The lemma is proved.

The conjugation condition (4.1) can be written in the form

$$\varphi_k(t) - \varphi^-(t) = \rho_k\overline{\varphi_k(t)} + f(t), \quad t \in L_k \ (k = 1, 2, \ldots, n). \tag{4.5}$$

A difference of functions analytic in D^+ and in D is in the left-hand part of the later relation. Then application of Sokhotskij's formulae yield

$$\varphi_k(z) = \sum_{m=1}^n \frac{\rho_m}{2\pi i}\int_{L_m}\frac{\overline{\varphi_m(t)}}{t-z}dt + f_k(z), \quad z \in D_k \ (k = 1, 2, \ldots, n), \tag{4.6}$$

where the function

$$f_k(z) = \frac{\lambda}{\pi i(\lambda_k + \lambda)}\sum_{m=1}^n\int_{L_m}\frac{f(t)}{t-z}\,dt$$

is analytic in D_k and Hölder continuous in its closure.

The integral equations (4.6) can be continued to L_k as follows:

$$\varphi_k(z) = \sum_{m=1}^{n} \rho_m \left[\frac{\overline{\varphi_k(z)}}{2} + \frac{1}{2\pi i} \int_{L_m} \frac{\overline{\varphi_m(t)}}{t-z} dt \right] + f_k(z),\ z \in L_k\ (k = 1, 2, \ldots, n). \tag{4.7}$$

One can consider equations (4.6), (4.7) as an equation with linear bounded operator in the space $\mathcal{H}(D^+)$.

Equations (4.6), (4.7) correspond to the generalized method of Schwarz. Write, for instance, equation (4.6) in the form

$$\varphi_k(z) - \frac{\rho_k}{2\pi i} \int_{L_k} \frac{\overline{\varphi_k(t)}}{t-z} dt = \sum_{m \neq k} \frac{\rho_m}{2\pi i} \int_{L_m} \frac{\overline{\varphi_m(t)}}{t-z} dt + f_k(z),\ z \in D_k\ (k = 1, 2, \ldots, n). \tag{4.8}$$

At the zeroth approximation we arrive at the problem for the single inclusion D_k $(k = 1, 2, \ldots, n)$,

$$\varphi_k(z) - \frac{\rho_k}{2\pi i} \int_{L_k} \frac{\overline{\varphi_k(t)}}{t-z} dt = f_k(z),\ z \in D_k. \tag{4.9}$$

Let problem (4.9) be solved. Further, its solution is substituted into the right-hand part of (4.8). Then we arrive at the first-order problem etc. Therefore, the generalized method of Schwarz can be considered as a method of implicit iterations applied to integral equations (4.6), (4.7).

In the case of circular domains the integral term from the left-hand part of (4.8) becomes a constant:

$$\frac{\rho_k}{2\pi i} \int_{L_k} \frac{\overline{\varphi_k(t)}}{t-z} dt = \rho_k \overline{\varphi_k(0)},$$

since $\overline{\varphi_k(t)}$ is analytically continued out of the circle L_k.

Remark 4.2. An integral equation method was proposed in Chapter 4 of [Mityushev and Rogosin (2000)] for the Dirichlet problem. A convergent direct iteration method for these equations coincides with the modified method of Schwarz. However, the integral terms of this method contain Green's functions of the domains D_k which should be constructed. One can obtain similar equations by application of the operator $\mathcal{S}_k^{-1}$ to both sides of (4.8), where the operator $\mathcal{S}_k$ solves equation (4.9).

4.2. Method of successive approximations

We use the following general result.

Theorem 4.3 ([Krasnosel'skii et al. (1969)]). *Let A be a linear bounded operator in a Banach space $\mathcal{B}$. If for any element $f \in \mathcal{B}$ and for any complex number ν satisfying the inequality $|\nu| \leq 1$ equation*

$$x = \nu A x + f \tag{4.10}$$

has a unique solution, then the unique solution of the equation

$$x = A x + f \tag{4.11}$$

can be found by the method of successive approximations. The approximations converge in $\mathcal{B}$ to the solution

$$x = \sum_{k=0}^{\infty} A^k f. \tag{4.12}$$

A weaker form of this theorem, valid for compact operators, is used in the proof of Lemma 2.2. Theorem 4.3 can be applied to equations (4.6), (4.7).

Theorem 4.4. *Let $|\rho_k| < 1$. Then the system of equations (4.6), (4.7) has a unique solution. This solution can be found by the method of successive approximations convergent in the space $\mathcal{H}(D^+)$.*

Proof. Let $|\nu| \le 1$. Consider equations in $\mathcal{H}(D^+)$,

$$\varphi_k(z) = \nu \sum_{m=1}^{n} \frac{\rho_m}{2\pi i} \int_{L_m} \frac{\overline{\varphi_m(t)}}{t-z} dt + f_k(z), \quad z \in D_k \ (k = 1, 2, \dots, n). \tag{4.13}$$

Equations on L_k look like (4.7).

Let $\varphi_k(z)$ be a solution of (4.13). Introduce the function

$$\varphi(t) = \varphi_k(t) - \nu \rho_k \overline{\varphi_k(t)} - f_k(t), \quad t \in L_k \ (k = 1, 2, \dots, n). \tag{4.14}$$

Calculate the integral

$$I = \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z} dt = \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{L_m} \frac{\varphi_m(t) - \nu \rho_m \overline{\varphi_m(t)} - f_m(t)}{t-z} dt, \ z \in D_k. \tag{4.15}$$

Taking into account (4.13), formulae

$$\frac{1}{2\pi i} \int_{L_m} \frac{\varphi_m(t)}{t-z} dt = 0, \ m \ne k, \ \frac{1}{2\pi i} \int_{L_k} \frac{\varphi_k(t)}{t-z} dt = \varphi_k(z), \ z \in D_k \tag{4.16}$$

and analogous formulae for $f_m(z)$, we obtain $I = 0$. The latter equality implies that $\varphi(t)$ is analytically continued into D. In accordance with Corollary to Theorem 1.1, the $\mathbb{R}$-linear problem (4.14) has a unique solution. This unique solution is the unique solution of the system (4.13).

Theorem 4.3 yields convergence of the method of successive approximations applied to the system (4.13).

This completes the proof of the theorem. □

5. Conclusion

Though the method of integral equations discussed in Section 4.2 is rather a numerical method, application of the residua for special shapes of the inclusions transforms the integral terms to compositions of the functions. Therefore, at least for the boundaries expressed by algebraic functions, one should arrive at the functional equations. An example concerning elliptical inclusions is presented in [Mityushev

(2009)]. This approach can be considered as a generalization of Grave's method reviewed in [Apel'tsin (2000)] to multiply connected domains.

In order to understand the place of the convergence results obtained in this paper, we return to Section 1.2. It was established in the previous works that for $|b(t)| < |a(t)|$ the problem has a unique solution. If the stronger condition (1.11) is fulfilled (always $S_p \geq 1$), this unique solution can be constructed by the absolutely convergent method of successive approximations. Absolute convergence implies geometrical restrictions on the geometry which can be roughly presented as follows. Each inclusion D_k is sufficiently far away from other inclusions D_m $(m \neq k)$. Only after the results presented in Section 4 of the present paper does the situation become clear and simplified. In the case (1.10) the method of successive approximations can be also applied, but absolute convergence is replaced by uniform convergence. The same story with convergence repeats for other methods and problems. In all previous works beginning from Poincaré's investigations, i.e., the Schwarz operator, the Poincaré series, the Riemann–Hilbert problem, the modified alternating Schwarz method etc., all the relevant problems were studied by absolute convergent methods under geometrical restrictions. The main result of the present paper is based on the modification of these methods and study of the problems by uniform convergence methods. This replacement of absolute convergence by uniform convergence abandons all previous geometrical restrictions and yields solution to the problems and convergence of the methods for an arbitrary location of non-overlapping inclusions.

This complicated situation concerning absolute and uniform convergences can be illustrated by a simple example. Let the almost uniformly convergent series $\sum_{n=1}^{\infty}(n-z)^{-2}$ $(z \notin \mathbb{N})$ be integrated term by term,

$$\int_w^z \sum_{n=1}^{\infty} \frac{1}{(n-t)^2}\,dt = \sum_{n=1}^{\infty}\left(\frac{1}{n-z} - \frac{1}{n-w}\right).$$

One can see that this series can be convergent if and only if $w \neq \infty$. This unlucky infinity is sometimes taken as a fixed point in similar investigations by specialists in complex analysis (see for instance Michlin's study [Mikhlin (1964)] devoted to convergence of Schwarz's method).

For engineers it is interesting to get exact and approximate formulae for the effective conductivity tensor. One can find a description of such formulae based on the solution to the problems discussed in the present paper in the survey [Mityushev et al. (2008)].

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