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OPTIMAL DISTRIBUTION OF THE NONOVERLAPPING CONDUCTING DISKS*

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Abstract. Conducting nonoverlapping identical disks are embedded in a two-dimensional background. The set of disks is infinite. The disks are distributed in such a way that the obtained composite is macroscopically isotropic. Let the conductivity of inclusions be higher than the conductivity of the matrix. It is proved that the hexagonal (triangular) lattice of disks possess the minimal effective conductivity when the concentration is not high.

Key words. effective conductivity, nonoverlapping disks, hexagonal lattice

AMS subject classifications. 74Q15, 30E25

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1. Introduction.

Conducting nonoverlapping identical disks of conductivity λ_1 are embedded in a two-dimensional background of conductivity λ. The set of disks is infinite. The disks are distributed in such a way that the obtained composite is macroscopically isotropic. Let $\widehat{\lambda}$ denote the ratio of the scalar effective conductivity to λ. It depends on the contrast parameter $\rho = \frac{\lambda_1 - \lambda}{\lambda_1 + \lambda}$, the concentration ν of the inclusions, and their location.

Physical hypothesis. The effective conductivity $\widehat{\lambda}$ attains the minimum for the regular hexagonal (triangular) array for any fixed ν and $\rho > 0$.

The hexagonal array (see Figure 2 or Figure 3(a)) is suggested as the optimal distribution due to the famous geometrical result by L. Fejes Tóth.

Geometrical theorem (see [12]). The densest packing of the equal disks in the plane is the hexagonal array. The density of this arrangement (concentration of inclusions) is equal to $\frac{\pi}{\sqrt{12}}$.

Kozlov [7] proposed an original method to study locations of inclusions for which the trace of the effective conductivity tensor attains the minimum. However, Kozlov's statement and solution should be revised and corrected, since his brilliant idea is not properly realized in [7].

In order to simplify the presentation we consider a macroscopically isotropic medium. The effective conductivity can be written in the form of the series [9]

$$\widehat{\lambda} = 1 + 2\rho\nu \left(1 + \sum_{k=1}^{\infty} A_k \nu^k\right), \tag{1}$$

where the coefficients A_k depend on the location of the disks. (See formulae (15)–(16) in the next section.) Actually, Kozlov [7] used the series (1) up to $O(\nu^6)$, i.e., investigated the coefficients A_1, A_2, A_3, A_4. These coefficients were calculated with the accuracy $O(\rho^2)$. However, A_3 and A_4 have higher order terms in ρ. (See formulae (16), where the terms with ρ^3 and ρ^4 are absent in Kozlov's paper.) Kozlov investigated "incomplete" coefficients A_3 and A_4, arrived at an optimization problem

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for the Weierstrass elliptic functions, and solved it. Why Kozlov lost some terms can be explained as follows. He used a generalization of the Rayleigh method to many inclusions in the periodicity cell. This method is based on the reduction of the conductivity problem to an infinite system of linear algebraic equations. This reduction implicitly contains a series in ρ. So, in order to obtain the proper coefficients A_3 and A_4 one must take a corresponding number of equations in the Rayleigh method.

It is more convenient to use the method of functional equations [9] where dependences on ν and ρ are explicitly presented. In the present paper, we follow these lines and reduce the optimal design problem to an optimization problem for the generalized Eisenstein–Rayleigh sums. This proves that the hexagonal array of disks attains the minimal $\widehat{\lambda}$ when the concentration is not high.

Remark 1. Macroscopically anisotropic composites attain the extremal conductivities for layered materials [1]. Hence, the problem of the optimal distribution of disks in this case is reduced to the packing problem of disks into layers.

The optimal problems for composites can be compared to the minimum problem for the Riesz s-energy $R_s = \sum_{i \neq j} |a_i - a_j|^{-s}$ $(s > 0)$ discussed in [11] and works cited therein. In the later problem, inclusions are replaced with point multipoles of order s. The parameters ν and ρ are absent in the Riesz s-energy. The optimal distribution of the points a_j in R_s can also be the hexagonal array [11].

2. Effective conductivity. Following [1, 4, 9], consider a doubly periodic composite. Let ω_1 and ω_2 be the fundamental pair of periods on the complex plane $\mathbb{C}$ such that $Im\ \frac{\omega_2}{\omega_1} > 0$. The fundamental parallelogram Q is defined by the vertices $\pm\frac{\omega_1}{2}$ and $\pm\frac{\omega_2}{2}$. Without loss of generality the area of Q can be normalized to one. The points $m_1\omega_1 + m_2\omega_2$ $(m_1, m_2 \in \mathbb{Z})$ generate a doubly periodic lattice $\mathcal{Q}$. Here, $\mathbb{Z}$ stands for the set of integer numbers.

Let i denote the imaginary unit. In the case $\omega_1 = \sqrt[4]{4/3}$ and $\omega_2 = \sqrt[4]{4/3}\, e^{\frac{\pi i}{3}}$, the cell Q becomes a rhombus with an angle $60°$ and the array $\mathcal{Q}$ is called the hexagonal lattice (the equilateral triangular lattice). In the cases $\omega_1 = 1$ and $\omega_2 = i$, the cell Q becomes a square and the array $\mathcal{Q}$ is called the square lattice. Only these two lattices generate isotropic structures.

Let the periods ω_1 and ω_2 be of order $\varepsilon \ll L$, where L is the linear order of the sample G bounded by a simple closed smooth curve Γ. Let $\Lambda_\varepsilon(z)$ be a measurable doubly periodic matrix-function satisfying the ellipticity condition [1, 4], and let f be a given continuous on Γ function. Consider the Dirichlet problem in the Sobolev space $H_0^1(G)$ [1, 4]

$$\nabla(\Lambda_\varepsilon(z)\nabla u_\varepsilon(z)) = 0, \tag{2}$$

$$u_\varepsilon(z) = f(z), \quad z \in \Gamma. \tag{3}$$

Let $\rightharpoonup$ denote the weak convergence in the space $L_2(G)$. Let

$$\Lambda_\varepsilon(z)\nabla u_\varepsilon(z) \rightharpoonup \widehat{\Lambda}\nabla u_0 \ \text{ in } L_2(G) \tag{4}$$

for some constant tensor $\widehat{\Lambda}$, where u_0 is a solution of the Dirichlet problem

$$\nabla(\widehat{\Lambda}\nabla u_0(z)) = 0, \quad u_0(z) = f(z), \quad z \in \Gamma. \tag{5}$$

Then, the tensor $\widehat{\Lambda}$ is called the effective conductivity tensor. The homogenization theory [1, 4] justifies the existence of the weak limit (4) and the independence of the limit of the shape of Γ and boundary conditions. For instance, instead of the Dirichlet

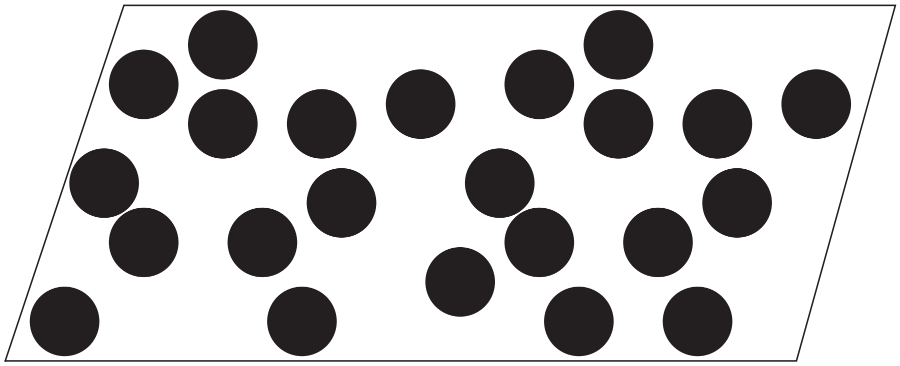

FIG. 1. N *disks in the periodicity cell* Q.

condition (3), the Neumann condition can be taken. The tensor $\widehat{\Lambda}$ for macroscopically isotropic composites has the form

$$\widehat{\Lambda} = \widehat{\lambda}\mathbf{I}, \tag{6}$$

where $\mathbf{I}$ is the identity tensor. The scalar $\widehat{\lambda}$ is called the effective conductivity.

3. Nonoverlapping circular disks. Consider N nonoverlapping circular disks D_k of radius r with the centers $a_k \in \mathbb{C}$ in the cell Q (see Figure 1). Let D_0 be the complement of all closure disks $|z - a_k| \leq r$ to the domain Q. We study conductivity of the doubly periodic composite when the host $\cup_{m_1,m_2}(D_0 + m_1\omega_1 + m_2\omega_2)$ and the inclusions $D_k + m_1\omega_1 + m_2\omega_2$ are occupied by materials of conductivities λ and λ_1, respectively ($m_1, m_2 \in \mathbb{Z}$). It is also assumed that the complex numbers a_k ($k = 1, 2, \ldots, N$) are distributed in Q in such a way that the corresponding two-dimensional composite is isotropic in macroscale, i.e., the effective conductivity of the composite is expressed by a scalar $\widehat{\lambda}$. The concentration of inclusions ν is defined as the ratio

$$\nu = \frac{1}{|Q|} \sum_{k=1}^{N} \pi r^2, \tag{7}$$

where $|Q|$ is the area of the periodicity cell.

We use the following terminology. A *lattice* (sublattice) always means a doubly periodic set of the points $m_1\omega_1 + m_2\omega_2$ ($m_1, m_2 \in \mathbb{Z}$) or a set of disks with the centers at these points. An *isotropic structure* is such a location of the disks in the cell Q that the effective conductivity of the considered composite is a scalar. For instance, a random distribution (see Figure 1) of N disks can generate an isotropic structure which is not a lattice in general. But the isotropic structure generated by seven disks in the dashed cell in Figure 2 can be considered as a lattice. An isotropic structure arranged in a lattice is called the isotropic lattice. Any isotropic lattice must be hexagonal or square.

Geometrical terms (packing, tessellation, etc.) are also used to supplement the main notations *lattice* and *isotropic structure*. It is worth noting that a lattice by definition is a partial case of the regular packings. For instance, a regular packing can be obtained from the hexagonal lattice by removing the centers of a hexagonal tessellation (see Figure 3). But the obtained set of the centers does not form a lattice since it cannot be generated by two translation vectors.

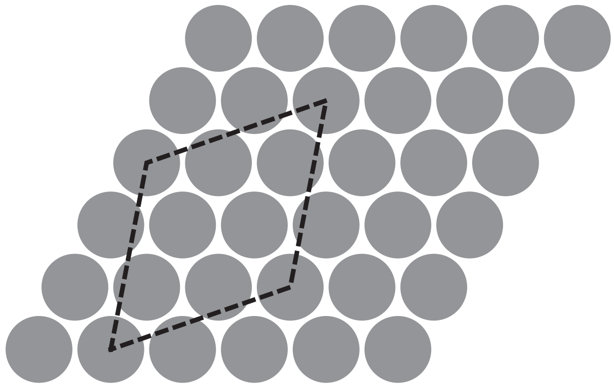

FIG. 2. *Seven disks in the dashed cell generate the hexagonal lattice.*

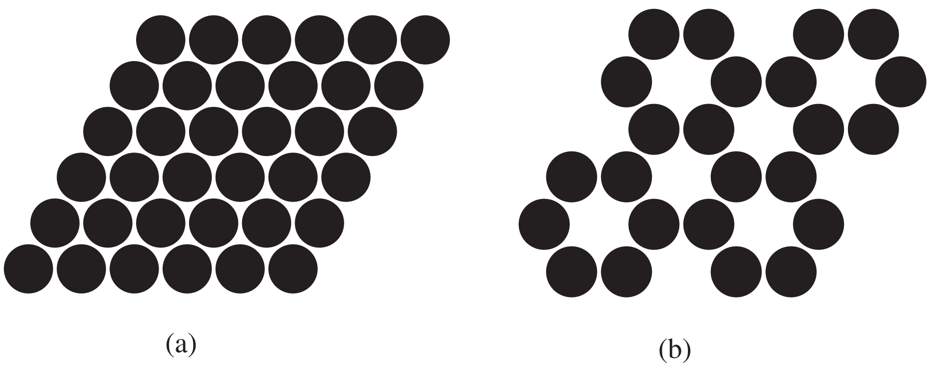

FIG. 3. (a) *Hexagonal lattice.* (b) *Regular packing obtained from the hexagonal lattice by removing some disks.*

In the considered optimization problem, the collection of possible locations of disks consists of the sets of doubly periodic locations with arbitrary ω_1, ω_2, and N (see Figure 1). The optimal distribution is hexagonal (see Figure 2 and Figure 3(a)) in accordance with the physical hypothesis.

In order to describe an approximate analytical formula for $\widehat{\lambda}$, we introduce the lattice sums [13, 9]

$$S_p = \sum_{(m_1,m_2)\neq(0,0)} \frac{1}{(m_1\omega_1 + m_2\omega_2)^p}, \quad p = 2, 3, \ldots, \tag{8}$$

and the Eisenstein functions [13, 9]

$$E_2(z) = \wp(z) + S_2, \quad E_p(z) = \frac{(-1)^p}{(p-1)!}\frac{d^{p-2}\wp(z)}{dz^{p-2}}, \quad p = 3, 4, \ldots, \tag{9}$$

where $\wp(z)$ is the Weierstrass function. The Eisenstein function $E_p(z)$ has a pole of order p at $z = 0$. However, it is convenient to define $E_p(0)$ as follows:

$$E_p(0) := S_p. \tag{10}$$

Instead of (8) the following formulae are used in computations [9]:

$$S_2=\left(\frac{\pi}{\omega_1}\right)^2\left(\frac{1}{3}-8\sum_{m=1}^{\infty}\frac{m\zeta^{2m}}{1-\zeta^{2m}}\right),\quad \zeta=\exp\left(\pi i\,\frac{\omega_2}{\omega_1}\right),\tag{11}$$

$$S_4=60\left(\frac{\pi}{\omega_1}\right)^4\left(\frac{4}{3}+320\sum_{m=1}^{\infty}\frac{m^3\zeta^{2m}}{1-\zeta^{2m}}\right),\tag{12}$$

$$S_6=1400\left(\frac{\pi}{\omega_1}\right)^6\left(\frac{8}{27}-\frac{448}{3}\sum_{m=1}^{\infty}\frac{m^5\zeta^{2m}}{1-\zeta^{2m}}\right).\tag{13}$$

The sums S_{2n} $(n\geq 4)$ are calculated by the recurrence formula

$$S_{2n}=\frac{3}{(2n+1)\,(2n-1)\,(n-3)}\sum_{m=2}^{n-2}(2m-1)\,(2n-2m-1)\,S_{2m}S_{2(n-m)},\tag{14}$$

$S_p=0$ for odd p. Perrins, McKenzie, and McPhedran [8] proved that $S_2=\pi$ for the hexagonal and square lattices. More precisely, they showed that $\tilde{S}_2=\frac{2\pi}{\sqrt{3}}$ but the area of their cell was equal to $|Q|=\frac{\sqrt{3}}{2}$, hence $S_2=\tilde{S}_2|Q|=\pi$.

Let q be a natural number; k_s runs over 1 to N and $m_s=2,3,\ldots$, where $s=1,2,\ldots,q$. Let $\mathbf{C}$ denote the operator of complex conjugation. The generalized Eisenstein–Rayleigh sum of order q is introduced by formula

$$e_{m_1\ldots m_q}:=\frac{1}{N^{1+\frac{m_1+\cdots+m_q}{2}}}\sum_{k_0k_1\ldots k_q}E_{m_1}(a_{k_0}-a_{k_1})\overline{E_{m_2}(a_{k_1}-a_{k_2})}\ldots\mathbf{C}^qE_{m_q}(a_{k_{q-1}}-a_{k_q}).\tag{15}$$

The effective conductivity can be calculated by (1), where $\nu=N\pi r^2$ is the concentration of the inclusions. The first few coefficients A_k have the form [9]

$$\begin{aligned}
&A_1=\frac{\rho}{\pi}e_2,\quad A_2=\frac{\rho^2}{\pi^2}e_{22},\ A_3=\frac{1}{\pi^3}\left[-2\rho^2e_{33}+\rho^3e_{222}\right],\\
&A_4=\frac{1}{\pi^4}\left[3\rho^2e_{44}-2\rho^3(e_{332}+e_{233})+\rho^4e_{2222}\right],\\
&A_5=\frac{1}{\pi^5}\left[-4\rho^2e_{55}+\rho^3(3e_{442}+6e_{343}+3e_{244})\right.\\
&\qquad\left.-2\rho^4(e_{3322}+e_{2332}+e_{2233})+\rho^5e_{22222}\right],\\
&A_6=\frac{1}{\pi^6}\left[5\rho^2e_{66}-4\rho^3(e_{255}+3e_{354}+3e_{453}+e_{552})\right.\\
&\qquad+\rho^4(3e_{2244}+6e_{2343}+4e_{3333}+3e_{2442}+6e_{3432}+3e_{4422})\\
&\qquad\left.-2\rho^5(e_{22233}+e_{22332}+e_{23322}+e_{33222})+\rho^6e_{222222})\right].
\end{aligned}\tag{16}$$

A simple iterative algorithm to calculate the next A_k is described in [9] and in papers cited therein. Below, only the coefficients A_1, A_2, A_3, A_4 are used to correct Kozlov's proof [7].

4. Generalized Eisenstein–Rayleigh sum for macroscopically isotropic composites. Formulae (16) are valid for arbitrary location of a_k $(k=1,2,\ldots,N)$. They can be simplified for macroscopically isotropic composites by relations between the generalized Eisenstein–Rayleigh sums.

Consider the normalized effective conductivity as a function $\widehat{\lambda} = \widehat{\lambda}(\rho)$ of ρ. It follows from Keller's identity [5] that

$$\widehat{\lambda}(\rho)\widehat{\lambda}(-\rho) = 1. \tag{17}$$

Substitute (1), (16) into (17) and compare the coefficients on ν^m $(m = 0, 1, \ldots, 5)$. Equations obtained by the coefficients on ν^0, ν^1, and ν^3 are identities. The coefficients on ν^2, ν^4, and ν^5 yield

$$e_2 = \pi, \quad e_{222} = 2\pi e_{22} - \pi^3, \quad e_{233} + e_{332} = 2\pi e_{33}. \tag{18}$$

This implies that the coefficients A_3 and A_4 from (16) can be simplified and the normalized effective conductivity $\widehat{\lambda}$ can be calculated up to $O(\nu^6)$ by formula

$$\widehat{\lambda} = \mu_0 + \mu + O(\nu^6). \tag{19}$$

The value $\mu_0 = 1 + 2\rho\nu + 2\rho^2\nu^2 - 2\rho^4\nu^4$ does not depend on the location of inclusions,

$$\mu = \frac{2\rho^3\nu^3}{\pi^2}\left[(1+2\rho\nu)e_{22} - \frac{2\nu}{\pi}(1+2\rho\nu)e_{33} + \frac{3\nu^2}{\pi^2}e_{44} + \frac{\rho^2\nu^2}{\pi^2}e_{2222}\right]. \tag{20}$$

Therefore, in order to study the dependence of $\widehat{\lambda}$ on the location of inclusions up to $O(\nu^6)$ it is sufficient to investigate the value (20). It is worth noting that formally μ is a complex number. However, we are going to show that μ is real due to the isotropy condition. More precisely, all the sums e_{22}, e_{33}, e_{44}, and e_{2222} are real.

Introduce the function

$$F_p(z) = \frac{1}{N}\sum_{k=1}^{N} E_p(z - a_k). \tag{21}$$

Then the sum e_{pp} for $p = 2, 4$ can be transformed as follows:

$$\begin{aligned} e_{pp} &= \frac{1}{N^p}\sum_{k_0,k_1} E_p(a_{k_0} - a_{k_1})\overline{F_p(a_{k_1})} = \frac{1}{N^{p-1}}\sum_{k_1} F_p(a_{k_1})\overline{F_p(a_{k_1})} \\ &= \frac{1}{N^{p+1}}\sum_{m=1}^{N}\left|\sum_{k=1}^{N} E_p(a_m - a_k)\right|^2. \end{aligned} \tag{22}$$

Here, the functions $E_2(z)$ and $E_4(z)$ are even. Along similar lines

$$e_{33} = -\frac{1}{N^4}\sum_{m=1}^{N}\left|\sum_{k=1}^{N} E_3(a_m - a_k)\right|^2. \tag{23}$$

Here, the function $E_3(z)$ is odd. On can see that $e_{pp} \geq 0$ for even p and $e_{pp} \leq 0$ for odd p. Similar arguments yield $e_{2222} \geq 0$. (See the next section.)

Let $\mathcal{Q}$ be a sublattice of $\mathcal{Q}'$. Then the fundamental vectors ω_1, ω_2 of $\mathcal{Q}$, and ω_1', ω_2' of $\mathcal{Q}'$ are related by equation

$$(\omega_1, \omega_2) = (\omega_1', \omega_2') \cdot A, \tag{24}$$

where the matrix A consists of the integer components and $\det A = N$ [13]. The factor $\mathcal{Q}/\mathcal{Q}'$ consists of N points $a_1, a_2, \ldots, a_N$ lying in the cell Q. These points also belong to $\mathcal{Q}'$, i.e., each a_j is a linear combination of ω_1' and ω_2' with integer coefficients. For instance, the hexagonal lattice generated by $\omega_1' = 1$, $\omega_2' = e^{\frac{\pi i}{3}}$ contains the hexagonal sublattice generated by $\omega_1 = 2\omega_1' + \omega_2'$, $\omega_2 = -\omega_1' + 3\omega_2'$ with $N = 7$ (see Figure 2).

Let $E_p(z; \omega_1, \omega_2)$ be the Eisenstein function of order p associated with the periods ω_1 and ω_2. Then [13]

$$\sum_{w \in \mathcal{Q}/\mathcal{Q}'} E_p\left(z + w; \omega_1, \omega_2\right) = E_p(z; \omega_1', \omega_2'). \tag{25}$$

In particular,

$$\sum_{w \in \mathcal{Q}/\mathcal{Q}'} E_p\left(w; \omega_1, \omega_2\right) = S_p(\omega_1', \omega_2'). \tag{26}$$

The area $|Q|$ of the fundamental cell Q holds unity and $|Q'| = N^{-1}$. Then change of the linear scale in (8) yields

$$S_p(\omega_1', \omega_2') = N^{\frac{p}{2}} S_p(\omega_1, \omega_2). \tag{27}$$

5. Minimal value of the effective conductivity. Let $J \subset \mathbb{C}^N$ denote the set of the centers $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ satisfying (18) and inequalities $|a_k - a_m| \geq 2r$ $(k \neq m)$. The conditions (18) are taken to restrict the study by isotropic structures. For definiteness it is assumed that $\rho > 0$, which is equivalent to inequality $\lambda_1 > \lambda$. In order to determine $\min_{J,Q} \widehat{\lambda}$ it is sufficient to investigate $\min_{J,Q} \mu(\mathbf{a})$, where $\mu = \mu(\mathbf{a})$ is given by (20) and considered as a function of $\mathbf{a}$ continuous and bounded on the compact subset J with fixed ν and ρ. Let $J^* \subset J$ denote the set of $\mathbf{a}$ generating an isotropic lattice. So, the points of J^* can form a lattice only of two types (square and hexagonal) of various sizes. The effective conductivity of the hexagonal lattice is less than the effective conductivity of the square lattice with fixed ν and $\rho > 0$ [8].

In order to minimize $\mu(\mathbf{a})$ consider the independent problems where the values e_{22}, $-e_{33}$, e_{44}, and e_{2222} are minimized. All these values attain the global minima not necessary at the same points of J. Let these minima be attained on the subsets J_{22}, J_{33}, J_{44}, and J_{2222} of the set J, respectively. If the set $\mathcal{J} = J_{22} \cap J_{33} \cap J_{44} \cap J_{2222}$ is not empty, then each $\mathbf{a} \in \mathcal{J}$ is a solution of the minimization problem for $\mu(\mathbf{a})$. The number N should not be fixed in this statement, which is natural in the theory of composites. Below we take such N that J contains a hexagonal lattice for appropriate ω_1 and ω_2.

LEMMA 2. *The set $\mathcal{J}$ contains at least the hexagonal lattice for which*

$$e_{22}^* = \min_{J,Q} e_{22} = \pi^2,\ -e_{33}^* = \min_{J,Q}(-e_{33}) = 0,\ e_{2222}^* = \min_{J,Q} e_{2222} = \pi^4, \tag{28}$$

$$e_{44}^* = \min_{J,Q} e_{44} = 0. \tag{29}$$

Proof. In order to determine e_{22}^* consider the function (21) with the parameter $\mathbf{a} \in J$ and $p = 2$ written as $F_2(z; \mathbf{a})$. The equality $e_2 = \pi$ is equivalent to the relation

$$\frac{1}{N} \sum_{m=1}^{N} F_2(a_m; \mathbf{a}) = \pi. \tag{30}$$

It follows from (22) that

$$e_{22} = \frac{1}{N} \sum_{m=1}^{N} |F_2(a_m; \mathbf{a})|^2. \tag{31}$$

Consider the inequality

$$\frac{1}{\pi^2 N}\sum_{m=1}^{N}|F_2(a_m;\mathbf{a})-\pi|^2 \geq 0,$$

which can be written in the form

$$\frac{1}{\pi^2 N}\sum_{m=1}^{N}\left\{|F_2(a_m;\mathbf{a})|^2-\pi[F_2(a_m;\mathbf{a})+\overline{F_2(a_m;\mathbf{a})}]+\pi^2\right\}\geq 0.$$

Application of (30)–(31) yields

$$e_{22}\geq \pi^2. \tag{32}$$

It follows from (26)–(27) with $p=2$ that the later inequality becomes equality for isotropic regular lattices. Therefore, the minimal value of e_{22} is attained, in particular, for the hexagonal lattice and equal to π^2. The first relation (28) is proved.

It follows from (23) that $-e_{33}\geq 0$. Consider the value of $-e_{33}$ on the set J^*. Application of (26) and $S_3=0$ yields zero. Therefore, the minimum is attained and $-e_{33}^*=0$.

Similar arguments can be applied to e_{44}. Application of (26) on the set J^* yields S_4, which is equal to zero for the hexagonal lattice. This proves (29).

In order to investigate e_{2222} introduce the function

$$G_2(z,\mathbf{a})=\frac{1}{N^2}\sum_{k_1,k_2}E_2(z-a_{k_1})\overline{E_2(a_{k_1}-a_{k_2})}. \tag{33}$$

Then the sum e_{2222} can be transformed similar to (22):

$$e_{2222}=\frac{1}{N}\sum_{k=1}^{N}|G_2(a_k,\mathbf{a})|^2. \tag{34}$$

Consider the inequality

$$\frac{1}{\pi^2 N}\sum_{m=1}^{N}|G_2(a_m;\mathbf{a})-\pi^2|^2\geq 0, \tag{35}$$

which can be written in the form

$$\frac{1}{\pi^2 N}\sum_{m=1}^{N}[|G_2(a_m;\mathbf{a})|^2-\pi^2(G_2(a_m;\mathbf{a})+\overline{G_2(a_m;\mathbf{a})})+\pi^4]\geq 0. \tag{36}$$

One can see that

$$\frac{1}{N}\sum_{m=1}^{N}G_2(a_m;\mathbf{a})=e_{22}. \tag{37}$$

Then (36) implies that

$$\frac{1}{\pi^2 N}\sum_{m=1}^{N}|G_2(a_m;\mathbf{a})|^2\geq 2\pi^2 e_{22}-\pi^4. \tag{38}$$

Using inequality $e_{22} \geq \pi^2$ we obtain

$$\frac{1}{\pi^2 N} \sum_{m=1}^{N} |G_2(a_m; \mathbf{a})|^2 \geq \pi^4. \tag{39}$$

This implies that $e^*_{2222} \geq \pi^4$. It follows from (26)–(27) that $e_{2222} = S_2^4$ when J coincides with J^*. For an isotropic structure $S_2 = \pi$, which yields $e^*_{2222} = \pi^4$.

The lemma is proved. ∎

LEMMA 3 (Kozlov [7]). *Let the Weierstrass function $\wp(z)$ be associated with the lattice $\mathcal{Q}$. The system of equations*

$$\begin{aligned} &\sum_{k \neq 1} \wp(a_k - a_1) = \sum_{k \neq 2} \wp(a_k - a_2) = \cdots = \sum_{k \neq N} \wp(a_k - a_N), \\ &\sum_{k \neq m} \wp'(a_k - a_m) = 0, \quad m = 1, 2, \ldots, N, \end{aligned} \tag{40}$$

with respect to $a_1, a_2, \ldots, a_N \in Q$ has exactly $N \prod_{p|N}(1 + \frac{1}{p})$ solutions up to translations (p is prime and divides N). Every solution $a_1, a_2, \ldots, a_N$ generates a lattice including the sublattice $\mathcal{Q}$.

THEOREM 4. *The minimal value of the normalized effective conductivity $\widehat{\lambda}$ up to $O(\nu^6)$ is attained only for the hexagonal lattice.*

Proof. In order to prove the theorem it is sufficient to show that the set $\mathcal{J}$ consists of only one element corresponding to the hexagonal lattice. Consider an element $\mathbf{a} \in \mathcal{J}$ for which $e_{22} = \pi^2$. It follows from (30)–(31) that

$$\frac{1}{\pi^2 N} \sum_{m=1}^{N} [|F_2(a_m; \mathbf{a})|^2 - \pi(F_2(a_m; \mathbf{a}) + \overline{F_2(a_m; \mathbf{a})}) + \pi^2] = 0. \tag{41}$$

The later equality is possible only if

$$F_2(a_m; \mathbf{a}) = \pi, \quad m = 1, 2, \ldots, N, \tag{42}$$

where in accordance with (21)

$$F_2(z; \mathbf{a}) = \frac{1}{N} \sum_{k=1}^{N} E_2(z - a_k). \tag{43}$$

Equality $e_{33} = 0$ for $\mathbf{a} \in \mathcal{J}$ and (23) imply that

$$\sum_{k=1}^{N} E_3(a_m - a_k) = 0, \quad m = 1, 2, \ldots, N. \tag{44}$$

The definition (43) for F_2 and equality $E_2'(z) = -2E_3(z)$ [13] yield

$$F_2'(a_m; \mathbf{a}) = 0, \quad m = 1, 2, \ldots, N, \tag{45}$$

where $F_2'(a_m; \mathbf{a}) = \frac{dF_2(z;\mathbf{a})}{dz}|_{z=a_m}$.

Consider now (42), (45) as a system of equations with respect to $\mathbf{a} \in \mathbb{C}^N$. It follows from Lemma 3 that every solution of this system forms a lattice including $\mathcal{Q}$

as a sublattice. This implies that $\mathbf{a}$ generates a lattice, i.e., $\mathbf{a} \in J^*$. The minimal value of $\mu(\mathbf{a})$ is attained only for the hexagonal lattice.

This proves the theorem. $\square$

Using the described method one can check that the next coefficients A_5 and A_6 from (16) attain the minimum for the hexagonal lattice. This implies that the accuracy in Theorem 4 can be increased up to $O(\nu^8)$. Investigations of all coefficients A_k could give the complete proof of the conjecture that the hexagonal lattice is optimal.

THEOREM 5. *The minimal value of the normalized effective conductivity* $\widehat{\lambda}$ *for sufficiently small* ρ *or* ν *is atained only for the hexagonal lattice.*

Proof. The proof is based on Theorem 4 and properties of series. Equation (1) yields

$$\widehat{\lambda} = \mu_0 + \mu_1 + \mu_2, \tag{46}$$

where $\mu_0 = 1 + 2\rho\nu + 2\rho^2\nu^2 - 2\rho^4\nu^4$,

$$\mu_1 = \frac{2\rho^3\nu^3}{\pi^2}\left[(1 + 2\rho\nu)e_{22} - \frac{2\nu}{\pi}e_{33}\right]. \tag{47}$$

The value $\mu_2 = O(\nu^5)$ is positive at least for sufficiently small ν, since the coefficient of ν^5 in (20) is positive. The global minimum μ_1^* of μ_1 is attained for the hexagonal lattice. Denote by μ_2^* the value of μ_2 for the hexagonal lattice. Assume that the global minimum $\tilde{\mu_1} + \tilde{\mu_2}$ of $\mu_1 + \mu_2$ is attained for other configurations, perhaps depending on ρ and on ν. Then

$$\mu_1^* + \mu_2^* > \tilde{\mu_1} + \tilde{\mu_2} \tag{48}$$

and simultaneously $\mu_1^* < \tilde{\mu_1}$ in accordance with Theorem 4. There exist such ν_0 that $\mu_2 < \frac{1}{2}(\tilde{\mu_1} - \mu_1^*)$ for all $\nu \le \nu_0$ and $\rho > 0$. Then $\mu_1^* + \mu_2^* < \mu_1^* + \frac{1}{2}(\tilde{\mu_1} - \mu_1^*) = \frac{1}{2}(\tilde{\mu_1} + \mu_1^*)$ and $\tilde{\mu_1} + \tilde{\mu_2} > \tilde{\mu_1} - \tilde{\mu_2} > \frac{1}{2}(\tilde{\mu_1} + \mu_1^*)$, which contradicts (48).

The same arguments can be applied for sufficiently small values of the parameter $\rho > 0$.

The theorem is proved. $\square$

Remark 6. Keller's identity (17) implies that if $\widehat{\lambda}$ attains the minimum for positive ρ, then the maximal value of $\widehat{\lambda}$ for negative ρ is attained for the same location of the disks.

6. Discussion. It is interesting to note that the solution to the physical problem of the minimal conductivity implies the solution to the geometrical problem of the optimal packing. In order to demonstrate this implication, assume that the physical hypothesis (see the introduction) is valid, i.e., the minimal conductivity for the fixed $0 < \nu < \frac{\pi}{\sqrt{12}}$ and $\rho = 1$ (disks are perfect conductors) is attained for the hexagonal lattice. Assume that there is the densest packing of the equal disks of radius r of the density $\nu_0 > \frac{\pi}{\sqrt{12}}$. Take the radius $r_0 < r$ of the eventual densest packing in such a way that the concentration ν becomes $\frac{\pi}{\sqrt{12}}$. The effective conductivity of the obtained structure is finite since the decreased disks do not touch. However, the effective conductivity of the hexagonal array is infinite because its disks are touching. This contradicts the minimal conductivity of the hexagonal array with fixed ν. Therefore, the densest packing is hexagonal.

Unfortunately, Theorem 5 cannot be used to justify the same implication for sufficiently small ρ and for $\nu = \nu_0$, since the structural approximation [2, 6] cannot be applied in this case as shown in [10].

Consider the optimal packings of the disks on the torus represented by a hexagonal lattice (on a triangular flat torus in the terminology of [3]) in the case $N = 2$. In order to be consistent with [3] consider the lattice generated by the fundamental vectors $\omega_1 = 1$ and $\omega_2 = e^{\frac{\pi i}{3}}$. Then the fundamental cell is the rhombus with the vertices 0, 1, $e^{\frac{\pi i}{3}}$, and $1 + e^{\frac{\pi i}{3}}$. The first isotropy condition $e_2 = \pi$ from (18) is reduced to equation $\wp(a_2) = 0$. The Weierstrass function has exactly two zeros located at the points $\frac{1}{2} + \frac{i\sqrt{3}}{6}$ and $1 + \frac{i}{\sqrt{3}}$. These points coincide with the points from [3] obtained by geometrical arguments without use of the Weierstrass function.

This remark shows that the physical problem of the effective conductivity and the geometrical problem of the optimal packing are closely connected and the optimal packing on torus can be determined by application of the elliptic functions.

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