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NEW BOUNDARY VALUE PROBLEMS AND THEIR APPLICATIONS TO INVISIBLE MATERIALS

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Two-dimensional conductivity problems with coated neutral inclusions are stated as boundary value problems for analytic functions. Their relations to eigenvalue problems, Courant's nodal domain theorem and non–linear Riemann–Hilbert problems are discussed. Solutions of the partial simple problems are given.

KEY WORDS: eigenvalue R-linear problem, neutral inclusion, boundary value problem, nodal set, non-linear Riemann-Hilbert problem
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1 INTRODUCTION

Mathematical models of invisibility are of considerable interest in a number of recent publications beginning with Kerker [10]. One can find the theoretical foundations and results devoted to various approaches of invisibility in [8], [12] and in the works cited therein. In this paper, we consider this problem in the context of conductivity of two-dimensional media with coated neutral inclusions. When the conductivity σ_0 of an isotropic matrix is chosen appropriately, one can insert a coated cylinder, with core conductivity σ_1 and coating conductivity σ, into the medium without disturbing the surrounding unidirectional external field. The previous works describe the effect of invisibility and contain examples of possibles coated inclusions. The paper [8] contains a general observation that any shaped inclusion with a smooth boundary can be made neutral by surrounding it with an appropriate coating. However, this assertion has not a rigorous mathematical justification in [8] and can be considered as a mathematical hypothesis.

In this paper, the results [8] and [12] are presented from mathematical point of view. Rigorous statements of the physical problems of invisibility yield new statements of the boundary value problems for analytic functions. The problems stated in Sec.2 describe neutral coatings for inclusions of finite conductivity [8]. Sec.3 concerns the limit cases of perfect conductors or insulators [12]. The problems stated in this paper have new features and are not investigated by mathematicians. Some of them recall the well known results by Schiffer [16] devoted to eigenvalues of the integral equations and by Courant [3] concerning nodal sets of the eigenfunctions. The paper [14] demonstrates that the eigenvalues of the considered problems can generate spaces of finite dimensions. The boundary value problems containing the non–linear conditions of type (12) were systematically discussed in [5, 17, 6, 7]. Thought various non–linear problems were investigated in the above works, non–linear boundary value problems combined with $\mathbb{R}$–linear condition (10) (see Sec.3) were not considered before.

2 EIGENVALUE R–LINEAR PROBLEM

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. Consider two positively oriented smooth simple curves Γ_1 and Γ_2 dividing $\hat{\mathbb{C}}$ onto three domains D_1, D and D_2, where the boundary of D_1 is Γ_1, the boundary of the doubly connected domain D consists of the curves $(-\Gamma_1)$ and Γ_2, D_2 contains infinity (see Fig.1).

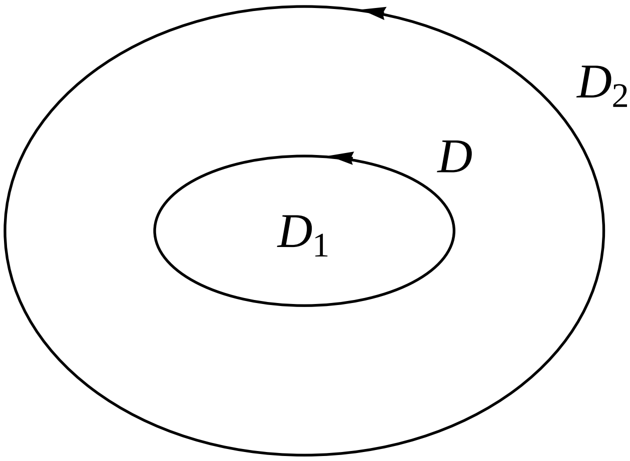

Рис. 1: Inclusion D_1, coating D and external medium D_2.

Problem 1. Given a real constant ρ. To find functions $\varphi_1(z)$, $\varphi(z)$, $\varphi_2(z)$ analytic in D_1, D, D_2, respectively, continuous in the closures of the considered domains and to find a complex constant $\lambda \neq 0$ such that the following $\mathbb{R}$–linear conditions are satisfied

$$\varphi(t) = \varphi_1(t) - \rho\overline{\varphi_1(t)}, \quad t \in \Gamma_1, \tag{1}$$

$$\varphi(t) = \varphi_2(t) - \lambda\overline{\varphi_2(t)}, \quad t \in \Gamma_2. \tag{2}$$

It is assumed that $\varphi_2(z)$ vanishes at infinity:

$$\varphi_2(\infty) = 0. \tag{3}$$

A non–zero function $\varphi_2(z)$ satisfying (1)–(3) is called the eigenfunction and the corresponding constant λ the eigenvalue of the problem. The function $\varphi_2(z)$ is distinguished from others, since it is the most important in applications. More precisely, let Γ_2 be the unit circle. Then the function $\omega(z) = \overline{\varphi_2\left(\frac{1}{\overline{z}}\right)}$ (if $\omega'(z) \neq 0$ in the unit disk) determines the external shapes of the inclusions $\omega(\Gamma_1)$ and of the corresponding neutral coatings $\omega(\Gamma_2)$.

The boundary condition (1) can be written in terms of the harmonic functions

$$u(z) = Re\,\varphi(z), \quad u_1(z) = \frac{2\sigma}{\sigma+\sigma_1} Re\,\varphi(z),\ z \in D_1 \tag{4}$$

as follows

$$u = u_1, \quad \sigma\frac{\partial u}{\partial n} = \sigma_1 \frac{\partial u_1}{\partial n} \text{ on } \Gamma_1. \tag{5}$$

Here, σ_1 and σ denote the conductivity coefficients related with ρ by formula

$$\rho = \frac{\sigma_1 - \sigma}{\sigma_1 + \sigma}. \tag{6}$$

Equations (5) model the perfect contact between two different materials [1, 2, 9]. Equation (2) can be written in the same form. Sometimes these equations are addressed to the spectral Poincaré–Steklov problems [11]. However, Poincaré and Steklov investigated other spectral problems of the type $u = k\frac{\partial u}{\partial n}$. Perhaps, separation of the boundary value problems and conjugation problems should be made. Then the first, who investigated a spectral problem for the $\mathbb{R}$–linear conditions, was Schiffer [16].

Our statement of the problem (1)–(3) differs from the statement of [16]. In the form (1)–(3), it is related to the neutral coating inclusions of the theory of composites [12], [8]. The main purpose of this section is to go further and to discuss the following

Conjecture 1. All eigenvalues are real (for real ρ). The set of eigenvalues is countable or finite. Let $|\lambda_1| \leqslant |\lambda_2| \leqslant \dots$. Then the corresponding eigenfunctions $\varphi_2^{(k)}(z)$ $(k=1,2,\dots)$ satisfy inequality

$$wind_{\Gamma_2}\varphi_2^{(k)}(z) \leqslant k, \tag{7}$$

where the winding number (or index [4]) is defined as

$$wind_{\Gamma_2} f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{f'(z)}{f(z)}\,dz.$$

Demonstration of Conjecture 1 for $k=1$ allows to justify that any shaped inclusion with a smooth boundary can be made neutral by surrounding it with an appropriate coating [8].

Consider the partial case of the problem (1)–(3) when $\Gamma_1=\{t\in \in\mathbb{C}: |t|=r\}$ and $\Gamma_2=\{t\in\mathbb{C}: |t|=1\}$ with $0<r<1$. All solutions of this problem have the following form

$$\varphi_1^{(k)}(z)=\alpha_k z^k, \quad \varphi_2^{(k)}(z)=-\overline{\alpha_k}\frac{\rho r^{2k}}{z^k}, \tag{8}$$

$$\varphi^{(k)}(z)=\alpha_k z^k-\overline{\alpha_k}\frac{\rho r^{2k}}{z^k}, \quad \lambda_k=\frac{1}{\rho r^{2k}}, \quad k\in\mathbb{Z},$$

where α_k are arbitrary complex numbers. It can be justified by direct substitution of the Laurent series of the unknown functions into (1)–(2) (see [13]). One can see that $wind_{\Gamma_2}\varphi_2(z)=k$, hence the functions $\varphi^{(k)}(z)$ satisfy Conjecture 1. Moreover, $\inf_k|\lambda_k|=|\lambda_1|$ and only the corresponding eigenfunction $\varphi_2^{(1)}(z)$ is conformal in $|z|>1$.

Conjecture 1 recalls Courant's theorem [3] outlined below. Consider for definiteness the Dirichlet problem $u=0$ on $\partial\Omega$ for equation $\Delta u=-\lambda u$ valid in a domain Ω. The set of eigenvalues consists of a sequence $0<\lambda_1\leqslant\lambda_2\leqslant\dots$ and the corresponding eigenfunctions $\varphi_1,\varphi_2,\dots$ constitute a complete orthonormal basis of $L_2(\Omega)$. The nodal set of a fixed φ_k is defined as the set $\{z \in \in \Omega : \varphi_k(z) = 0\}$. According to Courant's theorem the number of nodal domains of φ_k is less than or equal to k, for every $k = 1, 2, \dots$.

Conjecture 1 can be stated in terms of nodal domains of the eigenfunctions $Re\, \varphi_2^{(k)}(z)$ in D_2 of the problem (1)–(3). Instead of (7) one can demand that the number of nodal domains of $Re\, \varphi_2^{(k)}(z)$ is less than or equal to $2k$, for every $k = 1, 2, \dots$.

Let $\theta \in [0, 2\pi)$ denote the argument of the complex number z. It is easily seen that the nodal domains of the eigenfunctions $Re\, z^{-k} = = |z|^{-k} \cos k\theta$ from (8) are $2k$ sectors with the center at infinity divided by the rays $\theta = \frac{\pi m}{k}$ where $m = 0, 1, \dots, 2k-1$.

3 NON–LINEAR PROBLEM

Let G be a doubly connected domain on the complex plane of variable $\zeta = x + iy$ bounded by smooth simple curves Γ_1 and Γ_2 as in Sec.2 (write G instead of D and ζ instead of z). Let the curve Γ_1 is defined by equation

$$F(\zeta, \overline{\zeta}) = 0. \tag{9}$$

Problem 2. To find non–constant functions $\varphi(z)$, $\omega(z)$ analytic in $r < |z| < 1$, continuous in $r \leqslant |z| \leqslant 1$ and non–zero complex constants λ such that the $\mathbb{R}$–linear condition

$$\varphi(t) = \overline{\omega(t)} - \lambda \omega(t), \quad |t| = 1, \tag{10}$$

and the following boundary conditions hold

$$Im\, \varphi(t) = 0, \quad |t| = r, \tag{11}$$

$$F(\omega(t), \overline{\omega(t)}) = 0, \quad |t| = r. \tag{12}$$

Let $\zeta = \omega(z)$ denote the conformal mapping of $r < |z| < 1$ onto G. Then (12) means that $\omega(z)$ transforms the circle $|z| = r$ onto the curve Γ_1 defined by equation (9). In the theory of invisible inclusions, this conformal mapping determines Γ_1 by equations $\zeta = \omega(re^{2\pi i\theta})$ (given shape (9) of the inclusion) and $\zeta = \omega(e^{2\pi i\theta})$ (unknown shape of coating). Hence, it is interesting to consider the later problem with the additional condition that $\zeta = \omega(z)$ is a conformal mapping of D onto G, i.e.,

$$wind_{|t|=r}\omega(t) = wind_{|t|=1}\omega(t). \tag{13}$$

Example. Let the curve Γ_1 be the unit circle. Then $F(\zeta,\overline{\zeta}) = \zeta\overline{\zeta} - 1$. In this case, equation(12) becomes

$$\omega(t) = \frac{1}{\overline{\omega\left(\frac{r^2}{\overline{t}}\right)}}, \quad |t| = r. \tag{14}$$

Consider the problem when $\omega(z)$ is a conformal mapping. Then $\omega(z) \neq 0$ in $r \leqslant |z| \leqslant 1$ and equation (14) yields analytical continuation of $\omega(z)$ onto $0 < |z| < \infty$. Therefore, (14) is valid in $0 < |z| < \infty$.

One can see that

$$\lambda = -r^{-2}, \quad \omega(z) = cz, \quad \varphi(z) = \frac{\overline{c}}{z} - \lambda cz \tag{15}$$

is a solution of the problem (10), (11), (13), (14) for any complex constant c with $|c| = r^{-1}$.

Conjecture 2. All solutions of the problem (10), (11), (13), (14) have the form (15). This means that neutral circular inclusions can have only circular coatings.

We now proceed to discuss the partial problem (10), (11), (14) without (13). Since (14) is valid in $0 < |z| < \infty$, hence it can be used on the unit circle and substituted into (10)

$$\varphi(t) = \frac{1}{\omega(r^2 t)} - \lambda\omega(t), \quad |t| = 1. \tag{16}$$

It follows from the later relation that the function $\varphi(z)$ is analytically continued into $0 < |z| < \infty$ and

$$\varphi(z) = \frac{1}{\omega(r^2 z)} - \lambda\omega(z), \quad 0 < |z| < \infty. \tag{17}$$

The boundary condition (11) implies that

$$\varphi(z) = \overline{\varphi\left(\frac{r^2}{\overline{z}}\right)}, \quad 0 < |z| < \infty. \tag{18}$$

Substitution of (17) into (18) and simple transformations with use of (14) yield the functional equation

$$\lambda\omega(z) - \frac{\overline{\lambda}}{\omega(z)} = \omega\left(\frac{z}{r^2}\right) - \frac{1}{\omega(r^2 z)}, \quad 0 < |z| < \infty. \tag{19}$$

It is worth noting that all solution of the problem (10), (11), (14) satisfy (19), but not conversely because of the non–equivalent transformation of the problem to the functional equation.

One can check that $\omega(z) = cz^k$, $\lambda = r^{2k}$ $(k \in \mathbb{Z})$ with arbitrary constant c satisfy equation (19). It is not known, whether do exist other solutions. By appropriate normalization of c ($|c| = r^{-2k}$ by (14)) one can get a set of solutions of the problem (10), (11), (14).

It is interesting to investigate the problems (10)–(12) and (10)–(13) even in the case of algebraic functions $F(\zeta,\overline{\zeta})$ because any smooth curve can be approximated by algebraic curves.

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