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Schwarz-Christoffel Formula for Multiply Connected Domains

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Abstract. We derive a Schwarz-Christoffel formula for the conformal mapping of an arbitrary n-connected domain $\mathbb{D}$ bounded by mutually disjoint circles $|z-a_k| = r_k$, $k = 1, 2, \ldots, n$, onto the exterior of mutually disjoint polygons. The derivation is based on the exact solution to a Riemann-Hilbert problem for $\mathbb{D}$ without any geometric restriction imposed upon the location of the non-overlapping disks $|z-a_k| \leq r_k$.

Keywords. Multiply connected domain, Schwarz-Christoffel formula, Riemann-Hilbert problem, conformal mapping, functional equation, Poincaré series.

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1. Introduction

A Schwarz-Christoffel formula for multiply connected domains has recently been discussed by many mathematicians [1, 2, 3, 4, 5, 6, 7, 8, 9]. In order to describe the results consider a multiply connected circular domain $\mathbb{D}$ bounded by mutually disjoint circles $L_k = \{z \in \mathbb{C} \colon |z-a_k| = r_k\}$, $k = 1, 2, \ldots, n$, in the complex plane $\mathbb{C}$. The Schwarz-Christoffel formula was constructed as a conformal mapping of $\mathbb{D}$ onto a domain $\mathbb{P}$ bounded by n mutually disjoint polygons under the following geometric restriction to the locations of the circles [8]:

$$\max_{k \neq m} \frac{r_k + r_m}{|a_k - a_m|} < \frac{1}{(n-1)^{1/4}}. \tag{1}$$

Similar restrictions were imposed by many authors (see references in [11, 12, 13, 14, 15, 16]) to solve various boundary value problems in terms of absolutely convergent series including the alternating method of Schwarz. The Schottky-Klein prime function was used for such problems for arbitrary multiply connected

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domains [1, 2, 3, 4, 5]. However, explicit formulae for the Schottky-Klein prime function are known only under the restriction (1).

In the present paper, we follow the method presented in [15] which is based on the construction of the conformal mapping via exact solution of a Riemann-Hilbert boundary value problem [11, 13, 14, 16] and on the uniformly convergent Poincaré series [12] for the classical Schottky groups. As a result we explicitly obtain the Schwarz-Christoffel formula for arbitrary domains $\mathbb{D}$ without any geometric restriction. In order to construct the conformal mapping we solve the following Riemann-Hilbert problem [8]

$$\text{(2)} \qquad \operatorname{Re}[(t-a_k)\psi(t)] = -1, \qquad |t-a_k| = r_k,\ k = 1,2,\ldots,n,$$

for the function $\psi(z)$ analytic in $\mathbb{D}$ and continuous in $\mathbb{D}\cup\partial\mathbb{D}$ except at the point singularities on the boundary $\partial\mathbb{D}$ prescribed below.

2. Preliminaries

The clockwise orientation is taken on each L_k, hence the boundary $\partial\mathbb{D} = \bigcup_{k=1}^n L_k$ has the domain $\mathbb{D}$ to the left. First, we normalize the required conformal mapping $f\colon \mathbb{D}\to\mathbb{P}$ by the condition $f(\infty)=\infty$. Let the boundary of the domain $\mathbb{P}$ consist of n mutually disjoint polygons $\Gamma_k = f(L_k)$ with $\mathbb{P}$ lying to the left. Let the M_k vertices of Γ_k be denoted by $w_{\ell k}$, $\ell = 1,2,\ldots,M_k$, numbered clockwise around Γ_k for each $k = 1,2,\ldots,n$. The corresponding vertex angles of the Γ_k at the vertices $w_{\ell k}$, measured from the exterior of $\mathbb{P}$, are introduced as $\pi(1+\beta_{\ell k})$, where $\pi\beta_{\ell k}$ is the turning of the tangent at $w_{\ell k}$. The constants $\beta_{\ell k}$ satisfy the inequality $-1<\beta_{\ell k}\le 1$ and the relations

$$\text{(3)} \qquad \sum_{\ell=1}^{M_k}\beta_{\ell k} = 2, \qquad k = 1,2,\ldots,n.$$

The prevertices are denoted by $z_{\ell k}$ with $f(z_{\ell k}) = w_{\ell k}$.

Our study is based on the fact established in [8] that the preSchwarzian

$$\text{(4)} \qquad S(z) = \frac{f''(z)}{f'(z)}$$

is one of the solutions of the Riemann-Hilbert problem (2) in the class of functions having prescribed singularities at the points $z_{\ell k}\in L_k$, $\ell = 1,2,\ldots,M_k$, $k = 1,2,\ldots,n$, where

$$\text{(5)} \qquad S(z) \sim \frac{\beta_{\ell k}}{z - z_{\ell k}} \qquad \text{as } z\to z_{\ell k}.$$

In order to recover $f(z)$ from $S(z)$ one can integrate twice the relation (4) since $S(z) = (\ln f'(z))'$. The primitive function

$$\text{(6)} \qquad \omega(z) = \int^z S(\zeta)\,d\zeta$$

yields the Schwarz-Christoffel integral

$$f(z) = \int^z \exp(\omega(\zeta))\, d\zeta. \tag{7}$$

The preSchwarzian and $\exp(\omega(\zeta))$ were constructed by DeLillo et al. [8] by infinite sequences of iterated reflections that generate a series which absolutely converges under the restriction (1) (for details see [8] and the next section)

$$\exp(\omega(z)) = \prod_{m=1}^{n} \prod_{\ell=1}^{M_m} \left[\prod_{\gamma_o \in \mathcal{O}'_m} \frac{z - \gamma_o(\overline{z_{\ell m}})}{z - \gamma_o(\overline{a_m})} \prod_{\gamma_e \in \mathcal{E}'_m} \frac{z - \gamma_e(z_{\ell m})}{z - \gamma_e(a_m)} \right]^{\beta_{\ell m}}. \tag{8}$$

In the present paper, we follow another method based on the Riemann-Hilbert problem (2) in the class of functions satisfying the asymptotic formulae (5). General solution $\psi(z)$ of the problem (2) contains n arbitrary real constants [15], say $\xi_1, \ldots, \xi_n$, i.e. $\psi(z) = \psi(z; \xi_1, \ldots, \xi_n)$. In order to construct the required functions $S(z)$ and $\omega(z)$, we first construct the functions $\psi(z; \xi_1, \ldots, \xi_n)$ and $\Omega(z; \xi_1, \ldots, \xi_n)$. Further, the arbitrary constants $\xi_1, \ldots, \xi_n$ are chosen in such a way that $\psi(z; \xi_1, \ldots, \xi_n)$ yields $S(z)$ and $\Omega(z; \xi_1, \ldots, \xi_n)$ yields $\omega(z)$.

3. Schottky group

The inversion of z through the circle L_k is given by

$$z^*_{(k)} = \frac{{r_k}^2}{\overline{z - a_k}} + a_k.$$

It is known that if a function $\Phi(z)$ is analytic in the disk $|z - a_k| < r_k$ and continuous in its closure, then $\overline{\Phi(z^*_{(k)})}$ is analytic in $|z - a_k| > r_k$ and continuous in $|z - a_k| \geq r_k$.

Introduce the composition of successive inversions through the circles L_{k_1}, L_{k_2}, …, L_{k_p}

$$z^*_{(k_p k_{p-1} \ldots k_1)} := \left(z^*_{(k_{p-1} \ldots k_1)} \right)^*_{(k_p)}. \tag{9}$$

In the sequence $k_1, k_2, \ldots, k_p$ no two neighboring numbers are equal. The number p is called the level of the mapping. When p is even, these are Möbius transformations. If p is odd, we have anti-Möbius transformations, i.e., Möbius transformations in $\overline{z}$. Thus, these mappings can be written in the form

$$\gamma_j(z) = \frac{e_j z + b_j}{c_j z + d_j} \qquad \text{for } p \in 2\mathbb{Z}, \tag{10a}$$

$$\gamma_j(\overline{z}) = \frac{e_j \overline{z} + b_j}{c_j \overline{z} + d_j} \qquad \text{for } p \in 2\mathbb{Z} + 1, \tag{10b}$$

where $e_j d_j - b_j c_j = 1$, $j = 0, 1, 2, \ldots$. Here

$$\begin{aligned}
&\gamma_0(z) := z \\
&\qquad \text{(identical mapping with the level } p = 0), \\
&\gamma_1(\overline{z}) := z^*_{(1)}, \ldots, \gamma_n(\overline{z}) := z^*_{(n)} \\
&\qquad (n \text{ simple inversions, } p = 1), \\
&\gamma_{n+1}(z) := z^*_{(12)}, \gamma_{n+2}(z) := z^*_{(13)}, \ldots, \gamma_{n^2}(z) := z^*_{(n,n-1)} \\
&\qquad (n^2 - n \text{ double inversions, } p = 2), \\
&\gamma_{n^2+1}(\overline{z}) := z^*_{(121)}, \ldots \\
&\qquad \text{and so on.}
\end{aligned}$$

The set of the subscripts j of γ_j is ordered in such a way that the level p is increasing. The functions (10) generate a Schottky group $\mathcal{K}$. Thus, each element of $\mathcal{K}$ is presented in the form of the composition of inversions (9) or in the form of linearly ordered functions (10). Let $\mathcal{K}_m$ be a subset of $\mathcal{K}$ such that the last inversion of each element of $\mathcal{K}_m$ is different from $z^*_{(m)}$, i.e. $\mathcal{K}_m = \{z^*_{(k_p k_{p-1} \ldots k_1)} \colon k_p \neq m\}$. The set $\mathcal{K}'_m = \{z^*_{(k_p k_{p-1} \ldots k_1)} \colon k_1 \neq m\}$ is introduced similarly. All elements γ_j of the even levels generate a subgroup $\mathcal{E}$ of the group $\mathcal{K}$. The set of the elements γ_j of odd level $\mathcal{K} \backslash \mathcal{E}$ is denoted by $\mathcal{O}$. Introduce the notation $\mathcal{E}_m = \mathcal{E} \cap \mathcal{K}_m$, $\mathcal{O}_m = \mathcal{O} \cap \mathcal{K}_m$ and $\mathcal{E}'_m = \mathcal{E} \cap \mathcal{K}'_m$, $\mathcal{O}'_m = \mathcal{O} \cap \mathcal{K}'_m$.

Let us fix an inversion $z^*_{(m)}$. Consider the transformation

$$\gamma_j(z) = \left(\gamma_t^{-1}(\overline{z})\right)^*_{(m)}$$

from $\mathcal{E}$, where γ_t^{-1} is the inverse transformation to $\gamma_t \in \mathcal{O}'_m$. Then from [15] we have

$$\frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} = \frac{z - \gamma_t(\overline{\zeta^*_{(m)}})}{w - \gamma_t(\overline{\zeta^*_{(m)}})} \cdot \frac{w - \gamma_t(\overline{a_m})}{z - \gamma_t(\overline{a_m})}. \tag{11}$$

Consider now a transformation $\gamma_j \in \mathcal{O}$ and

$$\gamma_s(z) = \gamma_j^{-1}\left(\overline{z^*_{(m)}}\right)$$

from $\mathcal{E}'_m$. Then [15]

$$\overline{\left(\frac{\zeta - \gamma_j(\overline{w})}{\zeta - \gamma_j(\overline{z})}\right)} = \frac{w - \gamma_s(\zeta^*_{(m)})}{z - \gamma_s(\zeta^*_{(m)})} \cdot \frac{z - \gamma_s(a_m)}{w - \gamma_s(a_m)}. \tag{12}$$

4. Reduction of the Riemann-Hilbert problem to functional equations

It follows from [15] that the inhomogeneous Riemann-Hilbert problem (2) always has solutions. The general solution amounts of a particular solution of the inhomogeneous problem and a linear combination of n solutions of the homogeneous problem.

In order to solve the problem (2) rewrite it in the form of the $\mathbb{R}$-linear problem

$$(t-a_k)\psi(t) = (t-a_k)\psi_k(t) - \overline{(t-a_k)\psi_k(t)} - 1 + i\xi_k, \\ |t-a_k| = r_k,\ k = 1,\dots,n. \tag{13}$$

Here, ξ_k are undetermined real constants, $\psi_k(z)$ is analytic in $|z-a_k| < r_k$, continuous in $|z-a_k| \le r_k$ except the points $z_{\ell k}$, where

$$\psi_k(z) \sim \frac{\beta_{\ell k}}{2(z-z_{\ell k})} \qquad \text{as } z \to z_{\ell k}. \tag{14}$$

It will be shown below in Lemma 1 that the asymptotics (14) and (5) for $\psi(z)$ are matched. Hence, the $\mathbb{R}$-linear problem (13) must be stated in a class of functions with prescribed singularities. It is convenient to describe this class by introduction of the Banach spaces as follows.

Let G be a domain on the extended complex plane. Introduce the Banach space $\mathcal{C}(\partial G)$ of functions continuous on ∂G with the norm

$$\|F\| = \max_{t\in\partial G} |F(t)|.$$

Let us consider a closed subspace $\mathcal{C}_{\mathcal{A}}(G)$ of $\mathcal{C}(\partial G)$ consisting of functions analytically continued into G. The Maximum Principle implies that convergence in the space $\mathcal{C}_{\mathcal{A}}(G)$ is equivalent to uniform convergence in the closure of G. Let a fixed function F_0 have a finite number of singularities on the boundary ∂G. Introduce the space

$$\mathcal{C}_{\mathcal{A}}(G,F_0) = \{F\colon F - F_0 \in \mathcal{C}_{\mathcal{A}}(G)\}$$

of functions endowed with the norm

$$\|F\|_{\mathcal{C}_{\mathcal{A}}(G,F_0)} = \max_{t\in\partial G} |F(t) - F_0(t)|.$$

The spaces $\mathcal{C}_{\mathcal{A}}(G,F_0)$ and $\mathcal{C}_{\mathcal{A}}(G)$ are isomorphic.

Introduce mutually disjointed disks

$$\mathbb{D}_k = \{z \in \mathbb{C}\colon |z-a_k| < r_k\}, \qquad k = 1,2,\dots,n.$$

The multiply connected domain $\mathbb{D}$ complements all the closed disks $\mathbb{D}_k \cup L_k$ to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e.

$$\mathbb{D} = \widehat{\mathbb{C}} \backslash \bigcup_{k=1}^{n} (\mathbb{D}_k \cup L_k).$$

Introduce the function

$$\Phi(z) = \sum_{k=1}^{n} \Phi_k(z), \tag{15}$$

where

$$\Phi_k(z) = \frac{1}{2} \sum_{\ell=1}^{M_k} \frac{\beta_{\ell k}}{z - z_{\ell k}}. \tag{16}$$

Then $\psi(z) \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}, 2\Phi)$ and $\psi_k(z) \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}_k, \Phi_k)$. The problems (2) and (13) in the classes considered are equivalent in the sense of the following result.

Lemma 1.

(i) *If $\psi(z)$ and $\psi_k(z)$ are solutions of* (13) *in the class considered, then $\psi(z)$ satisfies* (2).
(ii) *If $\psi(z)$ is a solution of* (2), *there exist functions $\psi_k \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}_k, \Phi_k)$ and real constants ξ_k such that the $\mathbb{R}$-linear conditions* (13) *are fulfilled.*

Proof. The proof of the first assertion is evident. It is sufficient to take the real part of (13).

Conversely, let $\psi(z)$ satisfy (2). The function

$$\Psi_k(z) = \frac{i\xi_k}{2} + (z - a_k)\psi_k(z)$$

can be uniquely determined from the simple Schwarz problem for the disk $\mathbb{D}_k$ [10, 16]

$$2\,\mathrm{Im}\,\Psi_k(t) = \mathrm{Im}(t - a_k)\psi(t), \qquad |t - a_k| = r_k. \tag{17}$$

It is assumed that the function $\Psi_k(z)$ is continuous in $|z - a_k| \leq r_k$ except at the points $z_{\ell k}$, where the principal part $\beta_{\ell k}(z_{\ell k} - a_k)/[2(z - z_{\ell k})]$ of $\Psi_k(z)$ is determined by the right hand part of (17). The problem (17) for the function $\Psi_k(z)$ has a unique solution, since *Re* $\Psi_k(a_k) = 0$. Therefore, the function $\psi_k(z)$ and the constant ξ_k are uniquely determined in terms of $\psi(z)$ for each $k = 1, \ldots, n$. Direct calculations yields the asymptotic (14).

Hence the lemma is proved. ■

We now proceed to solve the $\mathbb{R}$-linear problem (13) written in the form

$$\psi(t) = \psi_k(t) - \left(\frac{r_k}{t - a_k}\right)^2 \overline{\psi_k(t)} - \frac{1 - i\xi_k}{t - a_k}, \qquad |t - a_k| = r_k,\ k = 1, \ldots, n. \tag{18}$$

Introduce the function

$$\tilde{\Phi}(z) := \begin{cases} \psi_k(z) + \sum_{m \neq k} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m(z^*_{(m)})} + \sum_{m \neq k} \frac{1 - i\xi_m}{z - a_m}, & |z - a_k| \leq r_k, \\ \psi(z) + \sum_{m=1}^{n} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m(z^*_{(m)})} + \sum_{m=1}^{n} \frac{1 - i\xi_m}{z - a_m}, & z \in \mathbb{D}. \end{cases}$$

Calculate its jump across the circle L_k

$$\Delta_k := \lim_{z \to t\, z \in \mathbb{D}} \tilde{\Phi}(z) - \lim_{z \to t\, z \in \mathbb{D}_k} \tilde{\Phi}(z), \qquad t \in L_k.$$

Using (18) we get $\Delta_k = 0$. It follows from the Analytic Continuation Principle that $\tilde{\Phi}(z)$ is analytic in the extended complex plane except at the points $z_{\ell k}$. A straightforward calculation shows that $\tilde{\Phi}(z)$ has the same asymptotics as $\Phi(z)$ from (15). Then the generalized Liouville theorem implies that $\tilde{\Phi}(z)$ coincides with $\Phi(z)$. Here, the relation $\tilde{\Phi}(\infty) = 0$ is used. The definition (15) of $\Phi(z)$ in $|z - a_k| \leq r_k$ yields the following system of functional equations

$$\psi_k(z) = \sum_{m \neq k} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} - \sum_{m \neq k} \frac{1 - i\xi_m}{z - a_m} + \Phi(z), \tag{19}$$
$$|z - a_k| \leq r_k,\ k = 1, \ldots, n.$$

The general solution of the Riemann-Hilbert problem (2) is constructed via $\psi_k(z)$ (see the definition of $\tilde{\Phi}(z)$ in $\mathbb{D}$)

$$\psi(z) = -\sum_{m=1}^{n} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} - \sum_{m=1}^{n} \frac{1 - i\xi_m}{z - a_m} + \Phi(z), \tag{20}$$
$$z \in \mathbb{D} \cup \partial\mathbb{D}.$$

The function $\psi(z)$ is analytic in $\mathbb{D}$ except at the points $z_{\ell k}$ where its principal part is $\beta_{\ell k}/(z - z_{\ell k})$.

5. Solution to functional equations

Lemma 2. *The system of functional equations* (19) *has a unique solution in* $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k, \Phi_k)$ *(*$k = 1, 2, \ldots, n$*). This solution can be found by the method of successive approximations.*

Proof. The proof of the lemma follows from [16, Lem. 4.8, p. 167] and [14] where functional equations had been solved in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$. Introduce the function χ_k analytic in $|z - a_k| < r_k$ and continuous in $|z - a_k| \leq r_k$ defined by

$$\chi_k(z) = \psi_k(z) - \Phi_k(z), \tag{21}$$

i.e. $\chi_k \in \mathcal{C}_A(\mathbb{D}_k)$. Substitution of (21) into (19) yields the system of functional equations in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$, $k = 1, 2, \ldots, n$,

$$\chi_k(z) = -\sum_{m\neq k} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\chi_m\left(z^*_{(m)}\right)} + h_k(z), \qquad |z-a_k| \le r_k,\ k = 1, \ldots, n, \tag{22}$$

where

$$h_k(z) = -\sum_{m\neq k} \frac{1-i\xi_m}{z-a_m} + \Phi(z) - \Phi_k(z) - \sum_{m\neq k} \left(\frac{r_m}{z-a_m}\right)^2 \overline{\Phi_m\left(z^*_{(m)}\right)} \tag{23}$$

belongs to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ (see (15) and (16)). It follows from [16, 14] that the system of functional equations (22) has a unique solution in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$. This solution $\chi_k(z)$ can be found by the method of uniformly convergent successive approximations. Then (21) yields $\psi_k(z) = \Phi_k(z) + \chi_k(z)$. Hence, $\psi_k(z)$ can be found by the method of successive approximations applied to (19). Convergence of the series for $\psi_k(z)$ is uniform in every compact subset of $\mathbb{D}_k \cup \partial\mathbb{D}_k \setminus \bigcup_{\ell=1}^{M_k}\{z_{\ell k}\}$.

This completes the proof of the lemma. ■

It is possible to write ψ_k explicitly in the form of a series. But ultimately a primitive of ψ_k is needed. In order to properly define it we fix a point $w \in \mathbb{D}\setminus\{\infty\}$ and introduce the functions

$$\varphi_m(z) = \int_{w^*_{(m)}}^{z} \psi_m(\zeta)\, d\zeta + \varphi_m(w^*_{(m)}), \qquad m = 1, 2, \ldots, n, \tag{24}$$

and

$$\Omega(z) = \sum_{m=1}^{n} \left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m\left(w^*_{(m)}\right)}\right] - \sum_{m=1}^{n} (1-i\xi_m) \ln\frac{a_m - z}{a_m - w} + \frac{1}{2}\sum_{m=1}^{n}\sum_{\ell=1}^{M_m} \beta_{\ell m} \ln\frac{z_{\ell m} - z}{z_{\ell m} - w}, \tag{25}$$

where a single valued branch of the logarithm is fixed in such a way that all cuts of

$$\ln\frac{a_m - z}{a_m - w}, \quad \ln\frac{z_{\ell m} - z}{z_{\ell m} - w}$$

lie in $\mathbb{D} \cup \mathbb{D}_m \cup \partial\mathbb{D}_m$ and $\ln x$ is real for positive $x \to +\infty$. In calculating the integral (24), the following relation is used [16]

$$\frac{d}{dz}\left[\overline{\varphi_m\left(z^*_{(m)}\right)}\right] = -\left(\frac{r_k}{z-a_k}\right)^2 \overline{\frac{d\varphi_m}{dz}\left(z^*_{(m)}\right)}, \qquad |z-a_k| > r_k. \tag{26}$$

The functions $\Omega(z)$ and $\varphi_m(z)$ belong to

$$\mathcal{C}_{\mathcal{A}}\left(\mathbb{D}, \sum_{m=1}^{n}\sum_{\ell=1}^{M_m}\beta_{\ell m}\ln(z-z_{\ell m})\right)$$

and to

$$\mathcal{C}_{\mathcal{A}}\left(\mathbb{D}_m, \frac{1}{2}\sum_{\ell=1}^{M_m}\beta_{\ell m}\ln(z-z_{m\ell})\right),$$

respectively. One can see from (24) that the function $\varphi_m(z)$ is determined by $\psi_m(z)$ up to an additive constant which vanishes in (25). The function $\Omega(z)$ vanishes at $z=w$.

Integrate each functional equation (19). Application of (24) yields functional equations

$$\varphi_k(z) = \sum_{m\neq k}\left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m\left(w^*_{(m)}\right)}\right] - \sum_{m\neq k}(1-i\xi_m)\ln\frac{a_m-z}{a_m-w}$$
$$+\frac{1}{2}\sum_{m=1}^{n}\sum_{\ell=1}^{M_m}\beta_{\ell m}\ln\frac{z-z_{\ell m}}{w-z_{\ell m}} + c_k, \qquad |z-a_k|\leq r_k,\ k=1,\dots,n, \tag{27}$$

for the functions

$$\varphi_k \in \mathcal{C}_{\mathcal{A}}\left(\mathbb{D}_k, \frac{1}{2}\sum_{\ell=1}^{M_k}\beta_{\ell k}\ln(z-z_{\ell k})\right)$$

and undetermined constants c_k.

Lemma 3. *The system of functional equations* (27) *with fixed* c_k *has a unique solution in*

$$\mathcal{C}_{\mathcal{A}}\left(\mathbb{D}_k, \frac{1}{2}\sum_{\ell=1}^{M_k}\beta_{\ell k}\ln(z-z_{\ell k})\right), \qquad k=1,\dots,n.$$

This solution can be found by the method of successive approximations.

Proof. The proof follows from Lemma 2, since (27) is the result of the integral operator

$$F \mapsto \int_{w^*_{(k)}}^{z} F(t)dt \tag{28}$$

applied to (19). Convergence in

$$\mathcal{C}_{\mathcal{A}}\left(\mathbb{D}_k, \frac{1}{2}\sum_{\ell=1}^{M_k}\beta_{\ell k}\ln(z-z_{\ell k})\right)$$

means uniform convergence in every compact subset of

$$(\mathbb{D}_k \cup \partial\mathbb{D}_k)\setminus \bigcup_{\ell=1}^{M_k}\{z_{\ell k}\}.$$

Therefore, the integral operator (28) can be applied term by term to the successive approximations for (19). This yields the uniformly convergent successive approximations for (28) in the compact subsets considered. ∎

Equations (27) can be compactly written in the operator form

$$X = AX + h, \tag{29}$$

where $X(z) = \varphi_k(z)$ in $|z - a_k| \leq r_k$, $k = 1, \dots, n$, the linear operator A and the function h are defined by the right hand part of (27). Application of Lemma 3 yields the representation for X in the form of uniformly convergent series

$$X = \sum_{s=0}^{\infty} A^s h. \tag{30}$$

Let

$$h = C_1 h_1 + C_2 h_2$$

with constants C_1, C_2 and

$$h_1, h_2 \in \mathcal{C}_{\mathcal{A}}\left(\mathbb{D}_k, \frac{1}{2}\sum_{\ell=1}^{M_k}\beta_{k\ell}\ln(z - z_{k\ell})\right) \qquad \text{for all } k = 1, \dots, n.$$

Then

$$X = X_1 + X_2, \qquad \text{where } X_1 = C_1\sum_{s=0}^{\infty} A^s h_1,\ X_2 = C_2\sum_{s=0}^{\infty} A^s h_2,$$

i.e.

$$\sum_{s=0}^{\infty} A^s h = C_1\sum_{s=0}^{\infty} A^s h_1 + C_2\sum_{s=0}^{\infty} A^s h_2. \tag{31}$$

The later equality in particular means that it is possible to change the order of summation in (30) in such a way that summation keeps the increasing level in each infinite sum. Uniqueness based on Lemma 3 yields the same results in the left and right parts of (31). Therefore, one can take any linear combination of $(1 - i\xi_m)\ln(z - a_m)$, $\beta_{mj}\ln(z - z_{mj})$ and compose the corresponding series for the solution. This observation allows us to avoid additional conditions related to unnecessary absolute convergence.

Applications of the successive approximations to (27) separately to the right hand part terms

$$c_k, \qquad \beta_{\ell m}\ln\frac{z - z_{\ell m}}{w - z_{\ell m}}, \qquad -\sum_{m\neq k}(1 - i\xi_m)\ln\frac{a_m - z}{a_m - w}$$

and summation of the results obtained (including $\frac{1}{2}\sum_{m=1}^{n}\sum_{\ell=1}^{M_m}$ for the second term) yields

$$
\begin{aligned}
\varphi_k(z) = c_k + \frac{1}{2}\sum_{m=1}^{n}\sum_{\ell=1}^{M_m}\beta_{\ell m}\Bigg(&\ln\frac{z_{\ell m}-z}{z_{\ell m}-w} + \sum_{k_1\neq k}\ln\overline{\frac{z_{\ell m}-z^*_{(k_1)}}{z_{\ell m}-w^*_{(k_1)}}} \\
&+\sum_{k_1\neq k}\sum_{k_2\neq k_1}\ln\frac{z_{\ell m}-z^*_{(k_2k_1)}}{z_{\ell m}-w^*_{(k_2k_1)}} \\
&+\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}\ln\overline{\frac{z_{\ell m}-z^*_{(k_3k_2k_1)}}{z_{\ell m}-w^*_{(k_3k_2k_1)}}}+\cdots\Bigg) \\
-\sum_{k_1\neq k}(1-i\xi_{k_1})\ln\frac{a_{k_1}-z}{a_{k_1}-w} &- \sum_{k_1\neq k}\sum_{k_2\neq k_1}(1-i\xi_{k_2})\ln\overline{\frac{a_{k_2}-z^*_{(k_1)}}{a_{k_2}-w^*_{(k_1)}}} \\
-\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}&(1-i\xi_{k_3})\ln\frac{a_{k_3}-z^*_{(k_2k_1)}}{a_{k_3}-w^*_{(k_2k_1)}} \\
-\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}\sum_{k_4\neq k_3}&(1-i\xi_{k_4})\ln\overline{\frac{a_{k_4}-z^*_{(k_3k_2k_1)}}{a_{k_4}-w^*_{(k_3k_2k_1)}}}+\cdots,
\end{aligned}
\tag{32}
$$

for $|z-a_k|\leq r_k$.

6. Construction of $f'(z)$

Substitute equation (32) for φ_k in (25) and write the result for the function $F(z;\xi_1,\xi_2,\ldots,\xi_n)=\exp(\Omega(z))$ in the form of the infinite product

$$
\begin{aligned}
F(z;\xi_1,\xi_2,\ldots,\xi_n) = \prod_{m=1}^{n}\prod_{\ell=1}^{M_m}\Bigg\{&\left(\frac{z_{\ell m}-z}{z_{\ell m}-w}\right)^{\beta_{\ell m}/2}\left[\prod_{k=1}^{n}\left(\overline{\frac{z_{\ell m}-z^*_{(k)}}{z_{\ell m}-w^*_{(k)}}}\right)^{\beta_{\ell m}/2}\right] \\
&\times\left[\prod_{k=1}^{n}\prod_{k_1\neq k}\left(\frac{z_{\ell m}-z^*_{(k_1k)}}{z_{\ell m}-w^*_{(k_1k)}}\right)^{\beta_{\ell m}/2}\right]\cdots\Bigg\} \\
\times\left[\prod_{k=1}^{n}\left(\frac{a_k-w}{a_k-z}\right)^{1-i\xi_k}\right]&\left[\prod_{k=1}^{n}\prod_{k_1\neq k}\left(\overline{\frac{a_{k_1}-w^*_{(k)}}{a_{k_1}-z^*_{(k)}}}\right)^{1-i\xi_{k_1}}\right] \\
\times\left[\prod_{k=1}^{n}\prod_{k_1\neq k}\prod_{k_2\neq k_1}\right.&\left.\left(\frac{a_{k_2}-w^*_{(k_1k)}}{a_{k_2}-z^*_{(k_1k)}}\right)^{1-i\xi_{k_2}}\right]\cdots.
\end{aligned}
\tag{33}
$$

This product converges uniformly in every compact subset of $\mathbb{D}\backslash\{\infty\}$.

Theorem 4. *The function* (33) *yields* $\exp(\omega(z))$ *from the Schwarz-Christoffel integral* (7) *when all* ξ_m *vanish. The function* $\exp(\omega(z))$ *has the form*

$$\begin{aligned}
(34)\qquad \exp(\omega(z)) &= F(z;0,\ldots,0)\\
&= \prod_{m=1}^{n}\prod_{j=1}^{M_m}\left\{\left(\frac{z_{\ell m}-z}{z_{\ell m}-w}\right)^{\beta_{\ell m}/2}\left[\prod_{k=1}^{n}\left(\frac{\overline{z_{\ell m}-z^*_{(k)}}}{\overline{z_{\ell m}-w^*_{(k)}}}\right)^{\beta_{\ell m}/2}\right]\right.\\
&\qquad\left.\times\left[\prod_{k=1}^{n}\prod_{k_1\neq k}\left(\frac{z_{\ell m}-z^*_{(k_1k)}}{z_{\ell m}-w^*_{(k_1k)}}\right)^{\beta_{\ell m}/2}\right]\cdots\right\}\\
&\quad\times\left(\prod_{k=1}^{n}\frac{a_k-w}{a_k-z}\right)\left(\prod_{k=1}^{n}\prod_{k_1\neq k}\frac{\overline{a_{k_1}-w^*_{(k)}}}{\overline{a_{k_1}-z^*_{(k)}}}\right)\\
&\quad\times\left(\prod_{k=1}^{n}\prod_{k_1\neq k}\prod_{k_2\neq k_1}\frac{a_{k_2}-w^*_{(k_1k)}}{a_{k_2}-z^*_{(k_1k)}}\right)\cdots .
\end{aligned}$$

Proof. The function $F(z;\xi_1,\xi_2,\ldots,\xi_n)$ is constructed on the basis of the general solution $\psi(z)$ of the Riemann-Hilbert problem (2) which contains the singularity function (4) as a particular solution. Therefore, $F(z;\xi_1,\xi_2,\ldots,\xi_n)$ contains the function $\exp(\omega(z))$ for appropriate constants $\xi_1,\xi_2,\ldots,\xi_n$. In order to prove the theorem it is sufficient to check that all singular points of the function (33) coincide to the singular points of the function $\exp(\omega(z))$ (including orders of the singularities) if and only if all ξ_m are equal to zero.

DeLillo et al. [8] used the Schwarz Reflection Principle to prove that the asymptotic behavior of the analytic continuation of $\exp(\omega(z))$ near the singular points is described by the relations

$$\exp(\omega(z)) \sim (z-\gamma_j(z_{\ell m}))^{\beta_{\ell m}} \qquad \text{as } z\to\gamma_j(z_{\ell m}), \tag{35a}$$

$$\exp(\omega(z)) \sim (z-\gamma_j(a_m))^{-2} \qquad \text{as } z\to a_m \tag{35b}$$

for $\gamma_j\in\mathcal{E}'_m$ and

$$\exp(\omega(z)) \sim (z-\gamma_j(\overline{z_{\ell m}}))^{\beta_{\ell m}} \qquad \text{as } z\to\gamma_j(\overline{z_{\ell m}}), \tag{36a}$$

$$\exp(\omega(z)) \sim (z-\gamma_j(\overline{a_m}))^{-2} \qquad \text{as } z\to a_m \tag{36b}$$

for $\gamma_j\in\mathcal{O}'_m$.

After application of the formula (3) one can observe all these singularities in (8). Though formula (8) does not hold in the general case, asymptotic formulae (35) and (36) always valid for $\exp(\omega(z))$ [8].

We now investigate the question, for which parameters $\xi_1,\xi_2,\ldots,\xi_n$ the function $F(z;\xi_1,\xi_2,\ldots,\xi_n)$ has the same asymptotic (35), (36). Let us fix a_m and consider the term

$$\mu(z)=\frac{z_{\ell k}-\gamma_e(z)}{z_{\ell k}-\gamma_e(w)}$$

from (33) where the last inversion of $\gamma_e \in \mathcal{E}$ is $z^*_{(m)}$, i.e. $\gamma_e(z)$ can be written as $\gamma_e(z) = z^*_{(k_p k_{p-1} \cdots k_1 m)}$ with odd p. The relation (11) implies that

$$\mu(z) = \frac{z - \gamma_t(\overline{(z_{\ell k})^*_{(m)}})}{w - \gamma_t(\overline{(z_{\ell k})^*_{(m)}})} \frac{w - \gamma_t(\overline{a_m})}{z - \gamma_t(\overline{a_m})}, \tag{37}$$

where $\gamma_t(\overline{z}) = z^*_{(k_1 k_2 \cdots k_p)}$ belongs to $\mathcal{O}'_m$. One can see that (37) gives the required first asymptotic (36) for $F(z; \xi_1, \xi_2, \ldots, \xi_n)$. It follows from (37) and (3) that the multiplier from (33)

$$\prod_{\ell=1}^{M_m} [\mu(z)]^{\beta_{\ell m}/2}$$

contains the multiplier

$$\prod_{\ell=1}^{M_m} [z - \gamma_t(\overline{a_m})]^{-\beta_{\ell m}/2} = [z - \gamma_t(\overline{a_m})]^{-1}.$$

Similar arguments can be applied to obtain the required first asymptotic (35) for $F(z; \xi_1, \xi_2, \ldots, \xi_n)$ by use of

$$\nu(z) = \frac{z_{\ell k} - \gamma_o(\overline{z})}{z_{\ell k} - \gamma_o(\overline{w})}$$

where $\gamma_o \in \mathcal{O}$. One can see also that the multiplier from (33)

$$\prod_{\ell=1}^{M_m} [\nu(z)]^{\beta_{\ell m}/2}$$

contains $[z - \gamma_s(a_m)]^{-1}$ for some $\gamma_s \in \mathcal{E}'_m$.

Therefore, the function $F(z; \xi_1, \xi_2, \ldots, \xi_n)$ contains the multiplier

$$[z - \gamma_s(a_m)]^{-2+i\xi_m}$$

for $\gamma_s \in \mathcal{E}'_m$ (the multiplier

$$[z - \gamma_t(\overline{a_m})]^{-2+i\xi_m}$$

for $\gamma_t \in \mathcal{O}'_m$) which determines its behavior near the singular point $z = \gamma_s(a_m)$ ($z = \gamma_t(\overline{a_m})$). One can see that $F(z; \xi_1, \xi_2, \ldots, \xi_n)$ has the same behavior as $\exp(\omega(z))$ at these points if and only if $\xi_m = 0$.

The theorem is proved. ■

Remark 1. It follows from Theorem 4 that

$$S(z) = \mathcal{O}(|z|^{-3}) \qquad \text{as } z \to \infty$$

(see (3) and (15)–(16)) that corresponds to [8].

We refer to the paper [15] for comparison of the general formula (34) with the partial formula (8) by DeLillo et al. [8]. A similar discussion concerning Riemann-Hilbert problems can be found in [11]–[14]. Here, we just note that (8) can be established by the limit $w \to \infty$ in (34), by using the arguments from [15] and from the proof of Theorem 4. It is worth noting that the substitution $w = \infty$ is forbidden in the general case since it yields the integral $\int_z^\infty \cdots$ in (25) after application of (24) and the substitution $\zeta = t^*_{(m)}$. This integral over the infinite path can produce a divergent series (for details see [15, Sect. 4] and [16, Sec. 4.10]).

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