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Neutral coated inclusions of finite conductivity

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We discuss the conductivity of two-dimensional media with coated neutral inclusions of finite conductivity. Such an inclusion, when inserted in a matrix, does not disturb the uniform external field. We are looking for shapes of the core and coating in terms of the conformal mapping $\omega(z)$ of the unit disc onto coated inclusions. The considered inverse problem is reduced to an eigenvalue problem for an integral equation containing singular integrals over a closed curve L_1 on the transformed complex plane. The conformal mapping $\omega(z)$ is constructed via eigenfunctions of the integral equation. For each fixed curve L_1, the boundary of the core is given by the curve $\omega(L_1)$. The boundary of the coating is obtained by the mapping of the unit circle. It is justified that any shaped inclusion with a smooth boundary can be made neutral by surrounding it with an appropriate coating. Shapes of the neutral inclusions are obtained in analytical form when L_1 is an ellipse.

Keywords: coated neutral inclusion; conformal mapping; boundary value problem

1. Introduction

Mathematical models of invisibility are of considerable interest in a number of recent publications. One can find the theoretical foundations and results devoted to various approaches of invisibility due to Kerker (1975), Ammari & Kang (2004, 2007), Alu & Engheta (2005), Milton & Nicorovici (2006), Milton *et al.* (2006), Farhat *et al.* (2008), Greenleaf *et al.* (2009), Guenneau *et al.* (2010), Liu (2010) and Ammari *et al.* (2011). In this paper, we investigate this problem in the context of conductivity of two-dimensional media with coated neutral inclusions. When the conductivity σ_0 of an isotropic matrix is chosen appropriately, one can insert a coated cylinder or sphere, with core conductivity σ_1 and coating conductivity σ, into the medium without disturbing the surrounding unidirectional external field. This effect is described in some detail by Milton (2000). Hashin (1962) constructed infinite packings of the plane by such coated cylinders of various sizes, also without disturbing the surrounding field. Hashin & Shtrikman (1962) established that the effective conductivity of such packings satisfy the famous Clausius–Mossoti approximation. These results were extended to coated ellipses and ellipsoids by Milton (1980, 1981). One can find an extended review with corresponding citations devoted to this problem in the book by Milton (2000) and in the paper by Milton & Serkov (2001).

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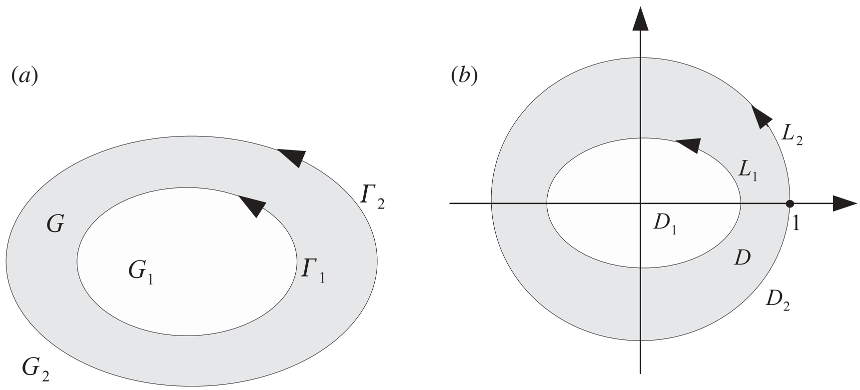

Figure 1. (*a*) Domains on the physical plane ζ. (*b*) Transformed domains on the plane z.

Milton & Serkov (2001) have posed the question of whether there are neutral coated inclusions other than single and multi-coated circles, spheres, ellipses and ellipsoids. By having used the assumption that the coating surrounds a hole or a perfect conductor, Milton & Serkov (2001) solved the problem for two-dimensional geometry. They discovered many possible shapes of the neutral inclusions. This result was obtained by use of the conformal mapping of the doubly connected coated domain onto a circular annulus. As a result, a boundary value problem for analytic functions arises. The latter problem was solved by the use of Laurent's series. It is worth noting that such problems were discussed by Schiffer (1959) and by Mityushev (1992) as well as Mityushev & Rogosin (1999) in the context of pure mathematical problems.

Our work is motivated by the question posed by Milton & Serkov (2001), which concerns a core of finite conductivity. Milton & Serkov (2001) noted that their analysis does not fit to this case, since they used a conformal mapping of the coating domain. Thus, the core domain cannot be taken into account apart from its boundary effects. Milton & Serkov (2001) tried to solve this problem approximately by small perturbations of the coated circles.

In this paper, the result of Milton & Serkov (2001) is extended to coated inclusions of finite conductivity represented in figure 1, where G_1, G, G_2 denote core, coating and the exterior domain; curve Γ_1 divides G_1 and G; curve Γ_2 divides G and G_2. We also apply the method of conformal mappings, but, contrary to Milton & Serkov (2001), a conformal mapping $\zeta \mapsto z$ of the simply connected domain bounded by the exterior coated boundary Γ_2 onto the unit disc is used. This conformal mapping yields a boundary value problem that does not have an easier form than the original one. However, the unknown boundary Γ_2 transforms onto the known unit circle.

One of the main results of this paper is that the neutral inclusion problem is reduced to an eigenvalue $\mathbb{R}$-linear problem on the auxiliary plane z (see §2*b*). In §3, the integral equation (3.12) corresponding to the eigenvalue problem is deduced. The solution of this problem depends on the curve L_1 dividing the transformed coated and core domains on the plane of variable z. It is worth noting that this eigenvalue problem has solutions for any Hölder continuous curve L_1. This observation implies that any shaped inclusion with a smooth boundary can be made neutral by surrounding it with an appropriate coating. Shapes of the neutral inclusions are obtained in analytical form in §§4 and 5 when L_1 is an ellipse.

2. Eigenvalue $\mathbb{R}$-linear problem

In this section, the problem of neutral inclusion is stated as an inverse boundary value problem for analytic functions with an unknown curve. This problem is reduced to an eigenvalue problem for an integral equation.

(*a*) *Statement of the problem and reduction to* $\mathbb{R}$*-linear problem*

Let the plane of complex variable $\zeta = x + iy$ be divided onto domains G_1, G, G_2 (figure 1) by simple closed smooth curves Γ_1 and Γ_2 oriented in counter-clockwise directions.

For definiteness, it is assumed that the origin belongs to G_1 and the domain G_2 contains the infinite point. The domains G_1 and G_2 are simply connected, G doubly connected; G_1, G and G_2 are, respectively, the core, coated and exterior domains occupied by materials with scalar conductivities σ_1, σ and σ_0. The potentials $u_1(x, y)$, $u(x, y)$ satisfy Laplace's equation in G_1, G and continuously differentiable in the closures of these domains. It is assumed that the potential $u_2(x, y)$ in the matrix G_2 is a linear function of the form

$$u_2(x, y) = -e_1 x - e_2 y, \tag{2.1}$$

where $\mathbf{e_0} = (e_1, e_2)$ is the uniform field outside the coated inclusion. Since $u_1(x, y)$ satisfies Laplace's equation in the simply connected domain G_1, it is represented as the real part of a function analytic in G_1

$$u_1(x, y) = \frac{2\sigma}{\sigma + \sigma_1} \operatorname{Re} \phi_1(\zeta), \quad \zeta \in G_1, \tag{2.2}$$

where the coefficient in (2.2) is introduced for convenience. The potential in the doubly connected domain G is represented in the form

$$u(x, y) = \operatorname{Re} \phi(\zeta) = \operatorname{Re} [\widetilde{\phi}(\zeta) + A \ln \zeta], \quad \zeta \in G, \tag{2.3}$$

where the function $\widetilde{\phi}(\zeta)$ is analytic and single-valued in G, and A is a real constant. It will be shown later that $A = 0$. Physically, this comes about because there is no net charge in the inclusion. The potential (2.1) can be also written in the complex form

$$u_2(x, y) = -\operatorname{Re} \overline{e_0}\zeta, \tag{2.4}$$

where the vector $\mathbf{e_0}$ is represented as the complex number $e_0 = e_1 + ie_2$; the bar denotes the complex conjugation.

Let $\mathbf{n}$ be the unit outward normal vector to the curve Γ_2, and let $\partial/\partial n$ be the outward normal derivative. Then the perfect contact between materials along the curve Γ_2 is described by equations

$$u = u_2, \quad \sigma\frac{\partial u}{\partial n} = \sigma_0\frac{\partial u_2}{\partial n} \quad \text{on } \Gamma_2. \tag{2.5}$$

Along similar lines

$$u = u_1, \quad \sigma\frac{\partial u}{\partial n} = \sigma_1\frac{\partial u_1}{\partial n} \quad \text{on } \varGamma_1. \tag{2.6}$$

Following Mityushev & Rogosin (1999), we transform the real conditions (2.5) and (2.6) into complex. Using (2.3) and (2.4), we rewrite the first relation (2.5) in the form

$$\operatorname{Re}\,\phi(\zeta) = -\operatorname{Re}\,\overline{e_0}\zeta, \quad \zeta \in \varGamma_2. \tag{2.7}$$

Let s be the natural parameter of the curve $\varGamma_2$ and $\partial/\partial s$ denote the corresponding derivative along $\varGamma_2$. Using the Cauchy–Riemann equation $\partial u_2/\partial n = \partial v_2/\partial s$, we rewrite the second equation (2.5) in the form of $\sigma \partial v/\partial s = \sigma_0 \partial v_2/\partial s$ on $\varGamma_2$ and integrate it along $\varGamma_2$ on s

$$\sigma v = \sigma_0 v_2 \quad \text{on } \varGamma_2. \tag{2.8}$$

The integration constant in (2.8) is taken to be zero, since the imaginary part of the complex potential is defined up to an arbitrary additive constant. It follows from (2.4) that the imaginary part $v_2(x, y) = -\operatorname{Im}\overline{e_0}\zeta$. Then (2.8) yields

$$\sigma \operatorname{Im}\phi(\zeta) = -\sigma_0 \operatorname{Im}\overline{e_0}\zeta, \quad \zeta \in \varGamma_2. \tag{2.9}$$

The two real equations (2.7) and (2.9) can be written in the form of one complex relation:

$$\phi(\zeta) = -\frac{\sigma_0 + \sigma}{2\sigma_0}\overline{e_0}\zeta - \frac{\sigma_0 - \sigma}{2\sigma_0}e_0\overline{\zeta}, \quad \zeta \in \varGamma_2. \tag{2.10}$$

The same arguments applied to (2.6) yield

$$\phi(\zeta) = \phi_1(\zeta) - \varrho\overline{\phi_1(\zeta)}, \quad \zeta \in \varGamma_1, \tag{2.11}$$

where the contrast Bergman parameter is introduced as follows:

$$\varrho = (\sigma_1 - \sigma)(\sigma_1 + \sigma)^{-1}. \tag{2.12}$$

This parameter satisfies inequality $|\varrho| < 1$ for finite positive σ_1 and σ.

In order to prove that $A = 0$ in the representation (2.3), we calculate the increment $[\phi]_{\varGamma_1}$ of the function $\phi(\zeta)$ along the closed curve $\varGamma_1$. It follows from (2.3) that $[\phi]_{\varGamma_1} = 2\pi\mathrm{i}A$. On the other hand, $[\phi]_{\varGamma_1}$ calculated by (2.11) yields zero. Hence, $A = 0$ in the representation (2.3) and $\widetilde{\phi}(\zeta) = \phi(\zeta)$ is a single valued function analytic in the doubly connected domain G.

Thus, we arrive at the boundary value problem (2.10)–(2.11) with unknown curve $\varGamma_2$ with respect to the complex potentials $\phi(\zeta)$, $\phi_1(\zeta)$ analytic in G, G_1 and continuously differentiable in the closures of the domains considered.

(b) *Reduction to eigenvalue problem*

The simply connected domain $G_1 \cup \varGamma_1 \cup G$ can be conformally mapped onto the unit disc U. Let

$$\zeta = -\frac{2\sigma_0}{(\sigma_0 - \sigma)\overline{e_0}}\omega(z) \tag{2.13}$$

denote the inverse conformal mapping of U onto $G_1 \cup \Gamma_1 \cup G$. The coefficient in (2.13) is introduced for convenience. Let this conformal mapping be normalized by the condition $\omega(0) = 0$. The unit circle L_2 transforms, as the boundary of U, onto the curve Γ_2, a simple closed curve $L_1 \subset U$ onto Γ_1 (figure 1).

The domains D_1 and D are transformed by (2.13) onto G_1 and G, respectively, the point $z = 0$ belongs to D_1. Introduce the complex potentials analytic in the considered domains D_1 and D

$$\varphi_1(z) = \phi_1(\zeta) \quad \text{and} \quad \varphi(z) = \phi(\zeta), \tag{2.14}$$

where the variables z and ζ are related by (2.13). The conjugation conditions (2.10) and (2.11) become

$$\varphi(t) = \overline{\omega(t)} - \lambda\omega(t), \quad |t| = 1 \tag{2.15}$$

and

$$\varphi(t) = \varphi_1(t) - \varrho\overline{\varphi_1(t)}, \quad t \in L_1, \tag{2.16}$$

where

$$\lambda = (\sigma + \sigma_0)(\sigma - \sigma_0)^{-1}. \tag{2.17}$$

One can see that λ^{-1} is equal to the second contrast parameter and $|\lambda| > 1$.

It is convenient to introduce the function $\varphi_2(z) := \overline{\omega(\overline{z}^{-1})}$ for $|z| \geq 1$ analytic in $|z| > 1$ and continuously differentiable in $|z| \geq 1$. The function $\varphi_2(z)$ vanishes at infinity, since $\omega(0) = 0$. Then (2.15)–(2.16) takes the form of the standard $\mathbb{R}$-linear condition (Mityushev & Rogosin 1999)

$$\varphi(t) = \varphi_2(t) - \lambda\overline{\varphi_2(t)}, \quad |t| = 1 \tag{2.18}$$

and

$$\varphi(t) = \varphi_1(t) - \varrho\overline{\varphi_1(t)}, \quad t \in L_1. \tag{2.19}$$

It is necessary to find a curve L_1 for which the problem (2.18)–(2.19) has a non-trivial solution. Then the non-zero function

$$\omega(z) = \overline{\varphi_2(\overline{z}^{-1})}, \quad |z| \leq 1 \tag{2.20}$$

can be a candidate for the conformal mapping. The function (2.20) besides being analytic must be a one-to-one map of U onto $G_1 \cup L_1 \cup G$.

Following Milton & Serkov (2001), it is more convenient to treat the above problem as an *eigenvalue problem*, with respect to the unknown constant λ and a fixed curve L_1 as follows: to find non-trivial functions $\varphi(z)$, $\varphi_1(z)$ and $\varphi_2(z)$ analytic in D, D_1 and D_2, respectively, and continuously differentiable in the closures of the domains considered with the $\mathbb{R}$-linear conjugation conditions (2.18) and (2.19). The constant λ has to be determined. Moreover, $\varphi_2(z)$ vanishes at infinity. The contrast parameter ϱ is supposed to be fixed in such a way that $|\varrho| < 1$.

3. Integral equation

In the present section, the $\mathbb{R}$-linear problem (2.18) and (2.19) is reduced to an integral equation. Let D_2 be the exterior of the unit circle. Let the curve L_1 and the unit circle L_2 be oriented in the counter-clockwise direction. Then the boundaries of the open domains D_1, D and D_2 are expressed by the relations $\partial D_1 = L_1$, $\partial D_2 = -L_2$ and $\partial D = L_2 \cup (-L_1)$. Let $\Psi^+(z)$ and $\Psi^-(z)$ be functions analytic in the domains D and $D^- = D_1 \cup D_2$, respectively, and Hölder continuous in their closures. Then, Cauchy's integral formula implies the following relations described by Gakhov (1966):

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{\Psi^+(t)}{t-z}\, \mathrm{d}t = \begin{cases} \Psi^+(z) & \text{for } z \in D \\ 0 & \text{for } z \in D^-, \end{cases} \tag{3.1a}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{L_2} \frac{\Psi^-(t)}{t-z}\, \mathrm{d}t = \begin{cases} \Psi^-(\infty) - \Psi^-(z) & \text{for } |z| > 1 \\ \Psi^-(\infty) & \text{for } |z| < 1 \end{cases} \tag{3.1b}$$

and

$$\frac{1}{2\pi \mathrm{i}} \int_{L_1} \frac{\Psi^-(t)}{t-z}\, \mathrm{d}t = \begin{cases} \Psi^-(z) & \text{for } z \in D_1 \\ 0 & \text{for } z \in D \cup L_1 \cup D_2. \end{cases} \tag{3.1c}$$

Let the function $\mu(t)$ be Hölder continuous on ∂D. The Cauchy-type integral

$$\Psi(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{\mu(t)}{t-z}\, \mathrm{d}t \tag{3.2}$$

represents a function analytic in the domains D, D^- and it satisfies the jump condition (Gakhov 1966)

$$\Psi^+(t) - \Psi^-(t) = \mu(t), \quad t \in \partial D, \tag{3.3}$$

where $\Psi^\pm(z)$ can be considered as the restriction of $\Psi(z)$ to $D \cup \partial D$ and to $D^- \cup \partial D$ or as the limit values of $\Psi(z)$ from the different sides of the curves ∂D. One can also consider formula (3.2) as the unique solution to the jump problem (3.3) with $\Psi^-(\infty) = 0$ due to Gakhov (1966).

The relations (2.18) and (2.19) with

$$\Psi^+(z) = \varphi(z), \quad z \in D; \quad \Psi^-(z) = \begin{cases} \varphi_1(z) & \text{if } z \in D_1 \\ \varphi_2(z) & \text{if } |z| > 1 \end{cases} \tag{3.4}$$

can be written in the form of the jump condition (3.3) with

$$\mu(t) = \begin{cases} -\lambda \overline{\varphi_2(t)} & \text{if } t \in L_2 \\ -\varrho \overline{\varphi_1(t)} & \text{if } t \in L_1. \end{cases} \tag{3.5}$$

Then (3.2) becomes the system of integral equations

$$\varphi_1(z) = -\frac{\lambda}{2\pi \mathrm{i}} \int_{|t|=1} \frac{\overline{\varphi_2(t)}}{t-z}\, \mathrm{d}t + \frac{\varrho}{2\pi \mathrm{i}} \int_{L_1} \frac{\overline{\varphi_1(t)}}{t-z}\, \mathrm{d}t, \quad z \in D_1, \tag{3.6}$$

and

$$\varphi_2(z) = -\frac{\lambda}{2\pi \mathrm{i}} \int_{|t|=1} \frac{\overline{\varphi_2(t)}}{t-z}\, \mathrm{d}t + \frac{\varrho}{2\pi \mathrm{i}} \int_{L_1} \frac{\overline{\varphi_1(t)}}{t-z}\, \mathrm{d}t, \quad |z| > 1. \tag{3.7}$$

The relation $t=\overline{t}^{-1}$ is fulfilled in the unit circle. The function $\overline{\varphi_2(\overline{t}^{-1})}$ is analytically continued into the unit disc, since $\varphi_2(z)$ is analytic in $|z|>1$. Then the first integral in (3.6) can be calculated by Cauchy's formula:

$$\frac{1}{2\pi i}\int_{|t|=1}\frac{\overline{\varphi_2(\overline{t}^{-1})}}{t-z}\,dt=\begin{cases}\overline{\varphi_2\left(\dfrac{1}{\overline{z}}\right)} & \text{if } z\in D_1\\ 0 & \text{if } |z|>1.\end{cases}\tag{3.8}$$

Substitution of (3.8) into (3.6) and (3.7) yields

$$\varphi_1(z)=-\lambda\overline{\varphi_2\left(\frac{1}{\overline{z}}\right)}+\frac{\varrho}{2\pi i}\int_{L_1}\frac{\overline{\varphi_1(t)}}{t-z}\,dt,\quad z\in D_1\tag{3.9}$$

and

$$\varphi_2(z)=\frac{\varrho}{2\pi i}\int_{L_1}\frac{\overline{\varphi_1(t)}}{t-z}\,dt,\quad |z|>1.\tag{3.10}$$

Substitute $\overline{z}^{-1}$ instead of z in (3.10) and take the complex conjugation

$$\overline{\varphi_2\left(\frac{1}{\overline{z}}\right)}=-\frac{\varrho}{2\pi i}\int_{L_1}\frac{\varphi_1(t)}{\overline{t}-(1/z)}\,d\overline{t},\quad |z|<1.\tag{3.11}$$

The function $\varphi_2(z)$ can be eliminated from the system (3.6)–(3.7), which is reduced to the integral equation

$$\varphi_1(z)=\varrho\left\{\frac{\lambda}{2\pi i}\int_{L_1}\frac{\varphi_1(t)}{\overline{t}-(1/z)}\,d\overline{t}+\frac{1}{2\pi i}\int_{L_1}\frac{\overline{\varphi_1(t)}}{t-z}\,dt\right\},\quad z\in D_1.\tag{3.12}$$

This integral equation has to be stated as the following eigenvalue problem. To find a non-zero function $\varphi_1(z)$ and a constant λ which satisfies equation (3.12). The function $\varphi_1(z)$ has to be analytic in D_1 and Hölder continuous in $D_1\cup L_1$.

Similar problems were already discussed by Bojarski (1960) and by Schiffer (1959), who considered the eigenvalue problem (3.6)–(3.7) when $\varrho=\lambda$. Schiffer (1959) proved that the eigenfunctions of that problem generate a complete orthogonal basis in a Hilbert space; all eigenvalues are real, generate a countable set and their moduli are greater than unity. This result does not fit to the considered case with fixed ϱ satisfying the inequality $|\varrho|<1$. However, it is possible to extend Schiffer's result to the problem (3.6)–(3.7) or to the equivalent problem (3.12). This result will be presented in a separate paper with necessary mathematical description of the functional space and singular integral operators.

In order to study the properties of the coatings and its dependence on the core, one can numerically solve the integral equation (3.12) for various L_1 and ϱ. Applying polynomial approximations for the function $\varphi_1(z)$, consider some examples of a numerical solution to (3.12). All the computations are performed for the normalized conductivity of the matrix $\sigma_0=1$. Let the shape of the core of conductivity $\sigma_1=3.36$ be given by the curve $\zeta=0.6e^{i\theta}+0.31e^{2i\theta}+0.07e^{3i\theta}+0.013e^{4i\theta}$ $(0\le\theta<2\pi)$. Three neutral coatings with different conductivity σ are presented in figure 2. It is interesting to note that the non-convex of the core have convex and non-convex exterior boundaries of the coatings.

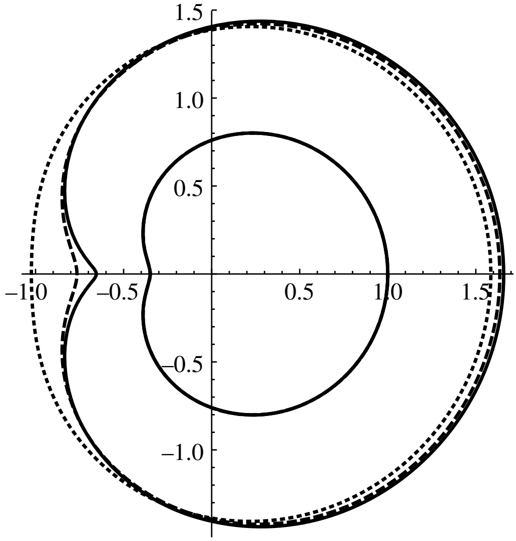

Figure 2. The fixed shape of the core of conductivity $\sigma_1 = 3.36$. Neutral shapes of coated inclusions for various conductivity: $\sigma = 1.64$ (solid line), $\sigma = 1.54$ (dashed line) and $\sigma = 1.32$ (dotted line).

Another example with the boundary curve of core $\zeta = 0.5e^{i\theta} + 0.35e^{2i\theta} + 0.12e^{3i\theta} + 0.06e^{4i\theta}$ $(0 \le \theta < 2\pi)$ is presented in figure 3. Here, we are looking for similar shapes of the coatings for different σ_1. All the found shapes differ locally near the point $z = -0.5$, where the boundary of coatings approaches to the core with increasing σ_1.

4. Functional equation

As noted at the end of §3, it is convenient to consider (3.12) as an eigenvalue problem with a fixed contour L_1. Each fixed contour L_1 on the plane z produces a neutral inclusion on the plane ζ. In order to constructively solve the integral equation (3.12), one can take an algebraic curve L_1 and reduce (3.12) to a functional equation. Any smooth curve can be approximated by an algebraic one. The simplest algebraic curve $L_1 = \partial D_1$ is a circle. One can check that the circle L_1 yields the circular annulus on the physical plane ζ described by Milton (2000). In the present section, D_1 is taken as an ellipse. This gives other non-trivial shapes of the neutral coated domain.

Let semi-axis of the ellipse L_1 be denoted by $r(1+\alpha)$ and $r(1-\alpha)$ $(0 < \alpha < 1$, $r > 0)$. The Joukowsky conformal mapping

$$z(w) = r\left(w + \frac{\alpha}{w}\right) \tag{4.1}$$

transforms the annulus $\sqrt{\alpha} < |w| < 1$ onto $D_1 \setminus \Gamma$, where Γ is the slit $(-2\sqrt{\alpha}, 2\sqrt{\alpha})$ along the x-axis (figure 4). The mapping (4.1) transforms D' onto D and D_2' onto D_2. The curve L_2' bounds a Bell's domain (Bell *et al.* 2009). The inverse

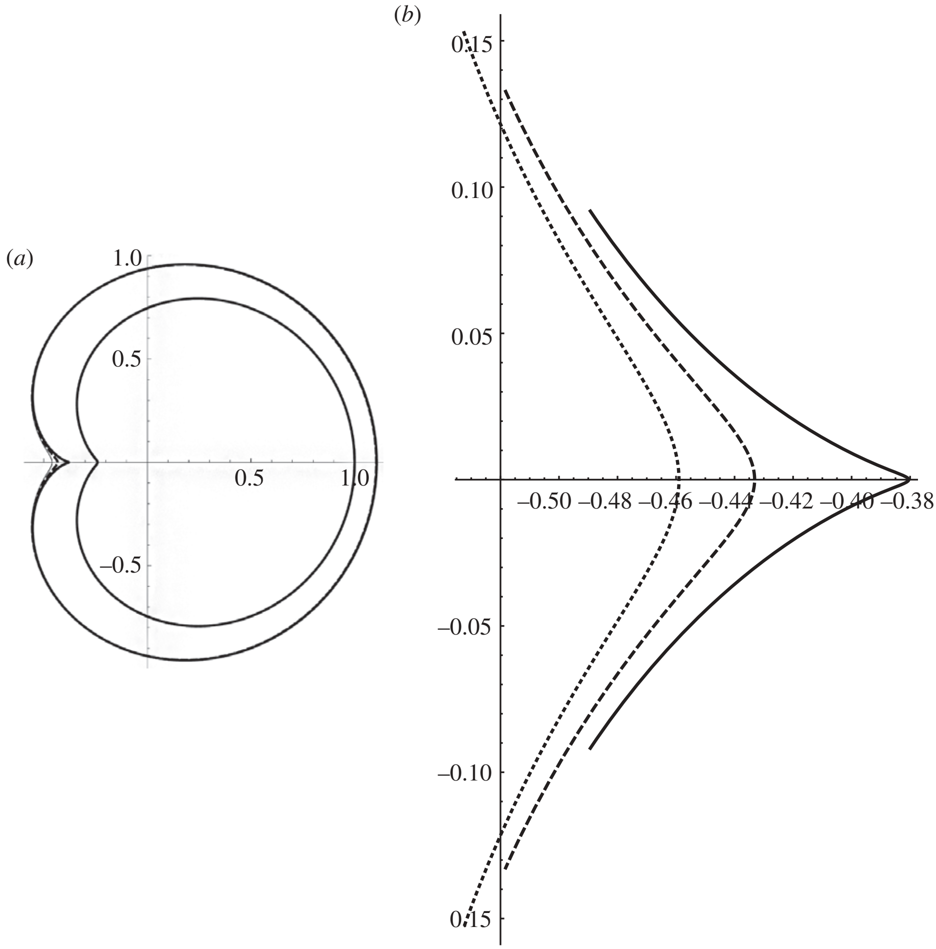

Figure 3. Closed neutral shapes of coated inclusions: (a) whole picture; (b) fragment near the point $z=-0.5$. For various conductivity of the core σ_1 conductivity of coating takes the corresponding σ: $\sigma_1=1.58$, $\sigma=172$ (solid line), $\sigma_1=2.79$, $\sigma=1.50$ (dashed line) and $\sigma_1=29.9$, $\sigma=1.18$ (dotted line).

mapping to (4.1) has the form

$$w(z)=\frac{1}{2}\left(\frac{z}{r}+\sqrt{\frac{z^2}{r^2}-4\alpha}\right), \tag{4.2}$$

where the branch of the root is chosen in such a way that $\lim_{z\to X\pm i0}\sqrt{(z^2/r^2)-4\alpha}=\pm i\sqrt{4\alpha-X^2}$, for $-2\sqrt{\alpha}r<X<2\sqrt{\alpha}r$.

Let τ run over the unit circle. Then $t=r(\tau+\alpha/\tau)$ runs over L_1. Introduce the function

$$\Phi_1(w)=\varphi_1\left(w+\frac{\alpha}{w}\right)=\varphi_1(z), \tag{4.3}$$

analytic in $\sqrt{\alpha}<|w|<1$, $(z\in D_1\setminus\Gamma)$ and Hölder continuous in $\sqrt{\alpha}\le|w|\le1$. It satisfies the condition

$$\Phi_1(w)=\Phi_1\left(\frac{\alpha}{w}\right),\quad |w|=\sqrt{\alpha}. \tag{4.4}$$

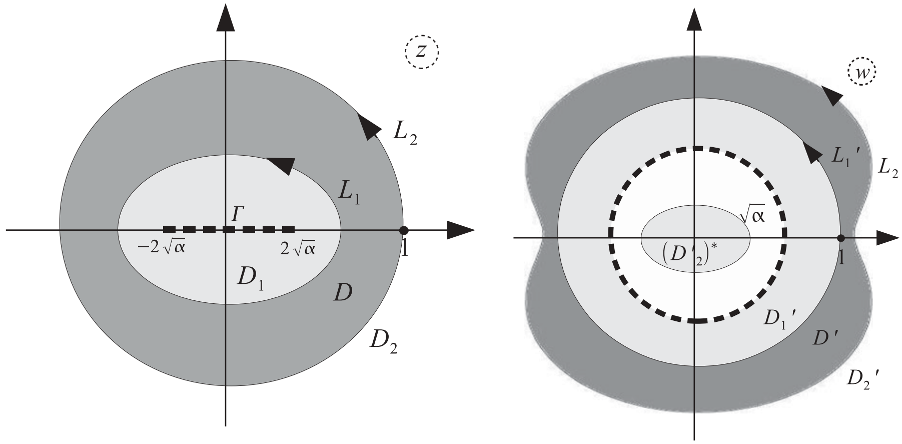

Figure 4. Transformation of the domains D_1 and D on the plane z onto D_1' and D' on the plane w. The domain $(D_2')^*$ is obtained from D_2' by the inversion $w \to 1/\bar{w}$ with respect to the unit circle.

Equation (4.4) implies that the function $\Phi_1(w)$ can be written in the form

$$\Phi_1(w) = \Phi(w) + \Phi\left(\frac{\alpha}{w}\right), \quad \sqrt{\alpha} \le |w| \le 1, \tag{4.5}$$

where $\Phi(w)$ is analytic in $|w| < 1$.

We now proceed to transform equations (3.9) and (3.10) on the plane z to deduce functional equations on the plane w. Similar method can be applied to (3.12) with the same result.

Let z belong to D_1. The integral from (3.9) becomes

$$\frac{1}{2\pi \mathrm{i}} \int_{L_1} \frac{\overline{\varphi_1(t)}}{t - z}\,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_{|\tau|=1} \frac{[\overline{\Phi(\tau)} + \overline{\Phi(\alpha/\tau)}](1 - \alpha/\tau^2)\,\mathrm{d}\tau}{\tau + (\alpha/\tau) - w - (\alpha/w)}, \tag{4.6}$$

where $z = r(w + \alpha/w)$, $|w| < 1$. The integral (4.6) can be calculated by residues. First, consider the integral

$$J_1 = \frac{1}{2\pi \mathrm{i}} \int_{|\tau|=1} \frac{\overline{\Phi(1/\bar{\tau})}(\tau^2 - \alpha)\,\mathrm{d}\tau}{\tau(\tau - w)(\tau - (\alpha/w))}, \tag{4.7}$$

where the function $\overline{\Phi(1/\bar{\tau})}$ is analytically continued into $|\tau| > 1$. It follows from inequality $\sqrt{\alpha} < |w| < 1$ that all roots of the denominator of (4.7) belong to the unit disc except at infinity. Hence,

$$J_1 = \frac{1}{2\pi \mathrm{i}} \int_{|\tau|=1} \frac{\overline{\Phi(1/\bar{\tau})}(\tau^2 - \alpha)\,\mathrm{d}\tau}{\tau(\tau - w)(\tau - (\alpha/w))} = \mathrm{res}_{\tau=\infty} = \overline{\Phi(0)}. \tag{4.8}$$

The second part of the integral (4.6) can be calculated by residues at the points $\tau = 0$; w; α/w lying in the unit disc:

$$J_2 = \frac{1}{2\pi \mathrm{i}} \int_{|\tau|=1} \frac{\overline{\Phi(\alpha\bar{\tau})}(\tau^2 - \alpha)\,\mathrm{d}\tau}{\tau(\tau - w)(\tau - (\alpha/w))} = -\overline{\Phi(0)} + \overline{\Phi(\alpha\overline{w})} + \overline{\Phi\left(\frac{\alpha^2}{\bar{w}}\right)}, \tag{4.9}$$

since the function $\overline{\Phi(\alpha\bar{\tau})}$ is analytic in $|\tau|<1$. The integral (4.6) is equal to J_1+J_2. Substitution of this result into (3.9) and the use of (4.3) and (4.5) yields

$$\Phi(w)+\Phi\left(\frac{\alpha}{w}\right)=-\overline{\varphi_2\left(\frac{1}{\bar{z}}\right)}+\varrho\left[\overline{\Phi(\alpha\overline{w})}+\overline{\Phi\left(\frac{\alpha^2}{\bar{w}}\right)}\right], \tag{4.10}$$

where $\sqrt{\alpha}\le|w|\le 1$.

Now we transform the integral (3.10) written in the form

$$\varphi_2(z)=\frac{1}{2\pi\mathrm{i}}\int_{|\tau|=1}\frac{[\overline{\Phi(1/\bar{\tau})}+\overline{\Phi(\alpha\bar{\tau})}](\tau^2-\alpha)\,\mathrm{d}\tau}{(\tau+(\alpha/\tau)-w-(\alpha/w))\tau^2}, \tag{4.11}$$

where $z=r(w+\alpha/w)$ with $|z|>1$, which implies that $|w|>1$. Calculate the first part of the integral (4.11) by residues in $|w|>1$

$$\frac{1}{2\pi\mathrm{i}}\int_{|\tau|=1}\frac{\overline{\Phi(1/\bar{\tau})}(\tau^2-\alpha)\,\mathrm{d}\tau}{\tau(\tau-w)(\tau-(\alpha/w))}=\mathrm{res}_{\tau=w}+\mathrm{res}_{\tau=\infty}=-\overline{\Phi\left(\frac{1}{\bar{w}}\right)}+\overline{\Phi(0)}. \tag{4.12}$$

The second part of the integral (4.11) is calculated by residues in $|w|<1$

$$\frac{1}{2\pi\mathrm{i}}\int_{|\tau|=1}\frac{\overline{\Phi(\alpha\bar{\tau})}(\tau^2-\alpha)\,\mathrm{d}\tau}{\tau(\tau-w)(\tau-(\alpha/w))}=\mathrm{res}_{\tau=0}+\mathrm{res}_{\tau=\alpha/w}=-\overline{\Phi(0)}+\overline{\Phi\left(\frac{\alpha^2}{\bar{w}}\right)}. \tag{4.13}$$

Substitution of (4.12) and (4.13) into (4.11) yields

$$\varphi_2(z)=\varrho\left[\overline{\Phi\left(\frac{\alpha^2}{\bar{w}}\right)}-\overline{\Phi\left(\frac{1}{\bar{w}}\right)}\right] \tag{4.14}$$

for z and w related by (4.1) and (4.2) when $|z|>1$ and $w\in D_2'$. Introduce the function $\Phi_2(w)$ analytic in D_2'

$$\Phi_2(w)=\varphi_2\left[r\left(w+\frac{\alpha}{w}\right)\right]. \tag{4.15}$$

The relation (4.15) can be written on the plane z

$$\varphi_2(z)=\Phi_2\left(\frac{1}{2}\left(\frac{z}{r}+\sqrt{\frac{z^2}{r^2}-4\alpha}\right)\right), \quad |z|\ge 1. \tag{4.16}$$

It follows from (4.16) that $\overline{\varphi_2(1/\bar{z})}=\overline{\Phi_2(\beta(\bar{w}))}$, where

$$\beta(\bar{w})=\frac{r+\sqrt{r^2-4\alpha(\bar{w}+(\alpha/\bar{w}))^2}}{2(\bar{w}+(\alpha/\bar{w}))}.$$

Therefore, equations (3.9) and (3.10) take the form

$$\left.\begin{aligned}\Phi(w)+\Phi\left(\frac{\alpha}{w}\right)&=-\lambda\overline{\Phi_2[\beta(\bar{w})]}+\varrho\left[\overline{\Phi(\alpha\bar{w})}+\overline{\Phi\left(\frac{\alpha^2}{\bar{w}}\right)}\right],\quad \sqrt{\alpha}\le|w|\le 1,\\ &\text{and}\\ \Phi_2(w)&=\varrho\left[\overline{\Phi\left(\frac{\alpha^2}{\bar{w}}\right)}-\overline{\Phi\left(\frac{1}{\bar{w}}\right)}\right],\quad |w|>1.\end{aligned}\right\}\tag{4.17}$$

Substitute Φ_2 from the second equation of (4.17) to the first one

$$\Phi(w)+\Phi\left(\frac{\alpha}{w}\right)=-\lambda\varrho\{\Phi[\alpha^2 f(w)]-\Phi[f(w)]\}+\varrho\left[\overline{\Phi(\alpha\bar{w})}+\overline{\Phi\left(\frac{\alpha^2}{\bar{w}}\right)}\right],\tag{4.18}$$

where

$$f(w)=[\overline{\beta(\bar{w})}]^{-1}=\frac{1}{2\alpha}\sum_{n=1}^{N}\frac{(2n)!r^{2(2n-1)}\alpha^n}{(2n-1)(n!)^2}\left(w+\frac{\alpha}{w}\right)^{2n-1}+O(r^{4N}).\tag{4.19}$$

This functional equation (4.18) can also be obtained from equation (3.12).

Introduce the operator P^+, which transforms Laurent's series to its regular part. More precisely, let $h(w)=\sum_{k=-\infty}^{+\infty}h_k w^k$, $\sqrt{\alpha}<|w|<1$. Then $(P^+h)(w)=\sum_{k=0}^{\infty}h_k w^k$, $|w|<1$. Application of P^+ to equation (4.18) yields

$$\Phi(w)=\varrho\overline{\Phi(\alpha\bar{w})}-\lambda\varrho P^+\{\Phi[\alpha^2 f(w)]-\Phi[f(w)]\},\quad |w|\le 1.\tag{4.20}$$

Thus, we arrive at the following eigenvalue problem. To find $\Phi(w)$ analytic in $|w|<1$ and continuous in $|w|\le 1$ with a constant λ satisfying equation (4.20).

5. Solution to functional equation

In this section, we solve the functional equation (4.20) using the polynomial approximation

$$\Phi(w)\approx\sum_{m=1}^{M}\alpha_m w^m\tag{5.1}$$

with a fixed number M. The operator from the right-hand part of (4.20) is compact in the Banach space endowed with the norm $||\Phi||=\max_{|w|\le 1}|\Phi(w)|$ (see Mityushev 1984). Therefore, approximate solutions of (4.20) can be found by replacement of (4.20) with equations of finite rank operators. Substitution of (5.1) into (4.20) and selection of the coefficients on w^m $(m=0,1,\dots,M)$ yields such an equation. Having used these arguments, Mityushev (1984) showed that the polynomial from (5.1) tends to $\Phi(w)$ as M tends to infinity.

We also use approximations on r^2 $(r<1)$. More precisely, all symbolic computations are performed with the accuracy $O(r^{2(2N-1)})$. For definiteness, the power $2(2N-1)$ is taken with an odd number $2N-1$. Substitute (5.1) and (4.19) in the functional equation (4.20), and then select the terms with the same powers w^m for $m=0,1,\ldots,M$. As a result, we obtain a system of homogeneous linear algebraic equations with respect to the coefficients α_m and the spectral parameter λ. This eigenvalue problem can be solved by a standard scheme. Substitution of the rational approximation from (4.19) into (4.20) instead of $f(w)$ is justified by the compactness of the operator containing $f(w)$. In order to illustrate the algorithm, we take $M=4$ and $N=8$. Then the system becomes

$$\left.\begin{aligned}
&-\alpha_1+r^2\lambda\varrho\alpha_1+\alpha\varrho\alpha_1-r^2\lambda\alpha^2\varrho\alpha_1+3r^6\lambda\alpha^2\varrho\alpha_1-3r^6\lambda\alpha^4\varrho\alpha_1\\
&\quad+3r^6\lambda\alpha\varrho\alpha_3-3r^6\lambda\alpha^7\varrho\alpha_3=0\\
&-\alpha_2+r^4\lambda\varrho\alpha_2+\alpha^2\varrho\alpha_2+8r^8\lambda\alpha^2\varrho\alpha_2-r^4\lambda\alpha^4\varrho\alpha_2-8r^8\lambda\alpha^6\varrho\alpha_2\\
&\quad+4r^8\lambda\alpha\varrho\alpha_4-4r^8\lambda\alpha^9\varrho\alpha_4=0\\
&r^6\lambda\alpha\varrho\alpha_1-r^6\lambda\alpha^3\varrho\alpha_1-\alpha_3+r^6\lambda\varrho\alpha_3+\alpha^3\varrho\alpha_3-r^6\lambda\alpha^6\varrho\alpha_3=0\\
&2r^8\lambda\alpha\varrho\alpha_2-2r^8\lambda\alpha^5\varrho\alpha_2-\alpha_4+r^8\lambda\varrho\alpha_4+\alpha^4\varrho\alpha_4-r^8\lambda\alpha^8\varrho\alpha_4=0.
\end{aligned}\right\}\tag{5.2}$$

Denote λ_j, $(j=1,\ldots,4)$ the eigenvalues of the system (5.2). For brevity, only the first eigenvalue λ is explicitly written

$$\begin{aligned}
\lambda_1=&[2r^8(-1+\alpha^2)^2(1+\alpha^2+\alpha^4)\varrho^2]^{-1}(r^2(-1+\alpha^2)\varrho(-1+\alpha^3\varrho+r^4\\
&\times(-1+\alpha(\varrho+\alpha(-4+\alpha(\varrho+\alpha(-1+4\alpha\varrho))))))-\surd(r^4(-1+\alpha^2)^2\varrho^2\\
&\times((-1+\alpha^3\varrho)^2+2r^4(-1+\alpha^2)(-1+\alpha^3\varrho)(-1+\alpha(\alpha+\varrho+2\alpha^2\varrho))\\
&+r^8(-1+\alpha(\varrho+\alpha(-4+\alpha(\varrho+\alpha(-1+4\alpha\varrho)))))^2))).
\end{aligned}\tag{5.3}$$

Each λ_j yields a non-zero solution of the system (5.2). If $\lambda=\lambda_1$, then α_1 is an arbitrary number, $\alpha_3=k_1\alpha_1$ and $\alpha_2=\alpha_4=0$, where

$$\begin{aligned}
k_1=&r^4\alpha[-1+\alpha\varrho+\alpha^3\varrho-\alpha^4\varrho^2-r^4(1-\alpha^2)](1-\alpha^3\varrho)^{-2}\\
&\times[(1-2\alpha\varrho-\alpha^3\varrho+2\alpha^4\varrho^2-\alpha^2(1-\varrho^2)].
\end{aligned}\tag{5.4}$$

If $\lambda=\lambda_2$, then α_2 is arbitrary, $\alpha_3=k_2\alpha_1$ and $\alpha_1=\alpha_3=0$. If $\lambda=\lambda_3$, then α_3 is arbitrary, $\alpha_3=k_3\alpha_1$ and $\alpha_2=\alpha_4=0$. If $\lambda=\lambda_4$, then α_4 is arbitrary, $\alpha_2=k_4\alpha_4$ and $\alpha_1=\alpha_3=0$. Coefficient k_j tends to zero as $r\to 0$. This means that $|k_j|$ is small for sufficiently small r. We do not write them all, because of their long form. The normalized eigenfunctions corresponding to the eigenvalues (5.3) have the form

$$\left.\begin{aligned}
&\Phi^{(1)}(w)\approx w+k_1w^3,\quad \Phi^{(2)}(w)\approx w^2+k_2w^3,\\
&\Phi^{(3)}(w)\approx k_3w+w^3\quad\text{and}\quad \Phi^{(4)}(w)\approx k_4w+w^4.
\end{aligned}\right\}\tag{5.5}$$

The map $\omega: U\to G_1\cup\Gamma_1\cup G$ is a one-to-one if and only if $\varphi_2(z)$ is a one-to-one map. The function $\varphi_2(z)$ is related to $\Phi(w)$ by (4.14). Therefore, $\varphi_2(z)$ is one-to-one in $|z|>1$ if and only if the function $F(w)=\Phi(\alpha^2w)-\Phi(w)$ is one-to-one in the domain $(D_2')^*$, the image of D_2' under the inversion with respect to unit circle (figure 4). The function $\Phi(w)$ is one for the functions (5.5). Consider the first function $\Phi^{(1)}(w)$. Then $F(w)=-w-k_1w^3+w\alpha+k_1w^3\alpha^3$. Let w_0 belong to

Table 1. Structure of the eigenvalues. Empty places correspond to zero.

	α_1	α_2	α_3	α_4	α_5	α_6
λ_1	1		k_{11}		k_{12}	
λ_2		1		k_{21}		k_{22}
λ_3	k_{31}		1		k_{32}	
λ_4		k_{41}		1		k_{42}
λ_5	k_{51}		k_{52}		1	
λ_6		k_{61}		k_{62}		1

$(D_2^*) \subset U$. Then equation $F(w_0) = F(w)$ with respect to w_0 has the following three roots:

$$w_1 = w_0, \quad w_{2,3} = -\frac{w_0}{2} \pm \frac{\sqrt{-(4 + 3k_1 w_0{}^2(1 + \alpha^2 + \alpha^4))}}{2\sqrt{k_1(1 + \alpha^2 + \alpha^4)}}. \tag{5.6}$$

For sufficiently small r, the roots w_2 and w_3 lie outside the unit disc. This implies that $F(w)$ is one-to-one mapping in $(D_2')^*$. Similar calculations can be done for $F(w)$ with $\Phi^{(j)}(w)$ $(j = 2, 3, 4)$ given by (5.5). In all these cases, $F(w)$ is not a one-to-one mapping.

In the case $M = 6$ and $N = 12$, we have six values of λ for which a system similar to (5.2) has non-zero solutions. The result is summarized in table 1.

For instance, $\alpha_1 = 1$, $\alpha_3 = k_{11}$, $\alpha_4 = k_{12}$, $\alpha_2 = \alpha_4 = \alpha_6 = 0$ if $\lambda = \lambda_1$. The coefficients k_{ij} are not written here because of their long form. We only note that $|k_{ij}|$ are sufficiently small for small r. This implies that the eigenfunction $\Phi^{(1)}(w) \approx w + k_{11}w^3 + k_{12}w^5$ is a one-to-one mapping.

For arbitrary even M and N $(N \geq 2M)$, we have M eigenvalues and eigenfunctions. Only the first function realizes one-to-one mapping. It can be approximately represented in the following form:

$$\Phi(w) \approx w + k_{N1}w^3 + \cdots + k_{N,M/2}w^M, \tag{5.7}$$

where $k_{N,j}$ tends to zero as $r \to 0$. The conformal mapping $\omega : U \to G_1 \cup \Gamma_1 \cup G$ is constructed via Φ by using (2.20) and (4.14):

$$\omega(z) = \varrho\left[\overline{\Phi\left(\frac{\alpha^2}{\overline{w(z)}}\right)} - \overline{\Phi\left(\frac{1}{\overline{w(z)}}\right)}\right], \tag{5.8}$$

where $w(z)$ is given by (4.2). Consider an example based on the approximation (5.5). Substitution of (5.5) into (5.8) yields $\omega(z) \approx z\varrho(1 + r^2z^2\alpha)(-1 + \alpha^2)$. Examples of computations are presented in figure 5.

6. Conclusion

The coated neutral inclusion problem for two-dimensional media is formulated as the eigenvalue $\mathbb{R}$-linear problem (2.18)–(2.19) on the auxiliary complex plane z. The later problem is reduced to the integral equation (3.12), where the closed curve L_1 can be arbitrarily fixed in the unit disc.

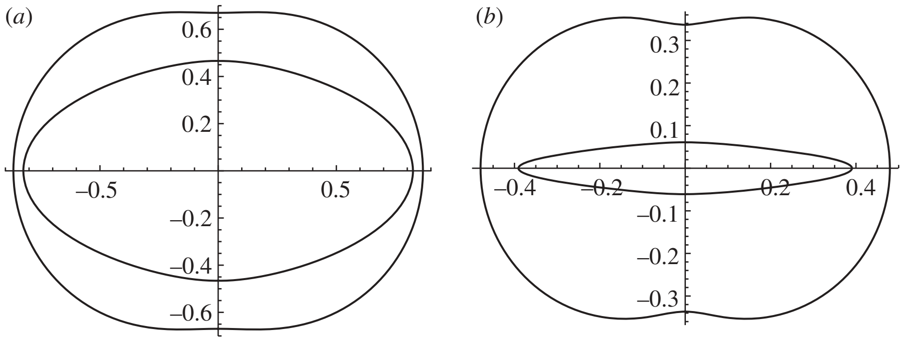

Figure 5. Neutral shapes of coated inclusions. (*a*) $\alpha = 0.2$, $\varrho = 0.9$ and $r = 0.8$, (*b*) $\alpha = 0.7$, $\varrho = 0.8$ and $r = 0.5$.

Each solution of the discussed eigenvalue problem can produce a solution for the neutral coated problem, if only the function $\omega(z)$ from (2.20) is a one-to-one map. Examples given in §5 show that the following conjecture can be posed. There exists at least one eigenfunction of the problem (3.9)–(3.10) or of (3.12) so that the corresponding function $\omega(z) = \overline{\varphi_2(1/\bar{z})}$ constructed by (3.11) is a one-to-one map of the unit disc onto $G_1 \cup \Gamma_1 \cup G_2$. This conjecture is equivalent to the existence of at least one eigenfunction from an infinite set of all eigenfunctions such that $\omega(z)$ in $|z| < 1$ has the unique zero at origin. Integral equation (3.12) gives a constructive algorithm to solve the neutral inclusion problem, since eigenvalues λ and eigenfunctions $\phi_1(z)$ can be computed by standard methods due to Krasnosel'skij *et al.* (1969). Then $\omega(z)$ is constructed by (2.20) and (3.11). Furthermore, zeros of $\omega(z)$ in the unit disc have to be investigated (see for instance Kravanja & Van Brel 2000). If $z = 0$ is the unique zero of $\omega(z)$ in the unit disc, the shape of the neutral inclusion is given by formula $\zeta = \omega(e^{i\theta})$, $0 \le \theta \le 2\pi$.

In the earlier mentioned algorithm, ϱ is a fixed number and λ is unknown. Using formulae (2.12) and (2.17), it is easy to state the eigenvalue problem in which conductivity σ has to be found. The spectral parameter λ is a function of ϱ, i.e. $\lambda = \lambda(\varrho)$ (see for instance (5.3) when L_1 is an ellipse). Then σ can be determined from equation $(\sigma + \sigma_0)/(\sigma - \sigma_0) = \lambda((\sigma_1 - \sigma)/(\sigma_1 + \sigma))$ with fixed σ_0 and σ_1.

It is difficult to precisely compare our results with Milton & Serkov (2001) since the $\mathbb{R}$-linear problem (2.18)–(2.19) does not coincide with the corresponding problem (3.17)–(3.18) from Milton & Serkov (2001) in the limit cases when $\sigma_1 = 0$ or $\sigma_1 = +\infty$. Moreover, it is difficult to match two results taken from two different infinite sets of all results. We can suggest that fig. 2(a) and (b) from Milton & Serkov (2001) can be obtained by using our approach when L_1 is an ellipse.

The results obtained in this paper allow us to make the following conclusion. Any two-dimensional core, i.e. a core of an arbitrary smooth shape and of an arbitrary conductivity, can be coated by such a material that the coated inclusion inserted in a matrix of an arbitrary fixed conductivity does not disturb the uniform field outside the inclusion. This conclusion can be justified by the following arguments. Any simply connected domain $G_1 \cup \Gamma \cup G$ can be conformally mapped onto the unit disc; L_1 in figure 4 is the image of Γ_1 in figure 1. For any Hölder continuous curve L_1, the integral equation (3.12) has non-trivial eigenfunctions. For sufficiently small ratios of the areas $|G_1|/(|G_1| + |G|)$ (that is equivalent to small r in §§4 and 5), one of the eigenfunctions produces the required conformal mapping $\omega(z)$. Therefore, this eigenfunction determines the shape of the coating Γ_2 (figure 1), and the corresponding eigenvalue determines the conductivity σ of the coating. It is not yet known whether the area of coating $|G|$ can be small, i.e. $|G_1|/(|G_1|+|G|)$ is of order 1. This question is related to the condition that $\omega(z)$ must be a one-to-one map. The earlier mentioned discussion is restricted to inclusions with smooth boundaries. The physical and geometrical properties of the coating inclusions can be systematically investigated via the integral equation (3.12). Some examples are discussed at the end of §3.

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