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Riemann-Hilbert Problems for Multiply Connected Domains and Circular Slit Maps

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Abstract. A conformal mapping of a multiply connected circular domain onto a complex plane with circular slits is obtained. No restriction on the location of the boundary circles is assumed. The mapping is derived in terms of the uniformly convergent Poincaré series by solution to a Riemann-Hilbert boundary value problem.

Keywords. Multiply connected domain, Riemann-Hilbert problem, functional equation, conformal mapping.

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1. Introduction

Many important functions of complex analysis such as harmonic measures, Green's function and others can be explicitly constructed by use of the Riemann-Hilbert problem for a multiply connected circular domain $\mathbb{D}$ bounded by mutually disjoint circles $L_k = \{z \in \mathbb{C}\colon |z - a_k| = r_k\}$, $k = 1, 2, \dots, n$, on the complex plane. This problem has applications in hydrodynamics [27], electrostatics [13], composites [14, 19] and other topics widely presented in [5, 6, 7, 15, 16]. The Riemann-Hilbert problem can be stated as a $\mathbb{C}$-linear problem on the Schottky double [30]. Such an interpretation relates the problem to the Abelian differentials and to other important objects on the Riemann surfaces.

Poincaré [26] introduced the θ_2-series associated to various types of Kleinian groups. He did not study Schottky groups carefully and just conjectured that the corresponding θ_2-series always diverges [26], [4, p. 51]. In 1891, Burnside [4] gave examples of convergent series for Schottky groups (which he called "the first class of groups") and studied their absolute convergence under some geometrical restrictions. In his study W. Burnside followed Poincaré's proof of the

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convergence of the θ_4-series. Burnside [4, p. 52] wrote "I have endeavoured to show that, in the case of the first class of groups, this series is convergent, but at present I have not obtained a general proof. I shall offer two partial proofs of convergency; one of which applies only to the case of Fuchsian groups, and for that case in general, while the other will also apply to Kleinian groups, but only when certain relations of inequality are satisfied." Further, Burnside [4, p. 57] gave a condition of the absolute convergence in terms of the coefficients of the Möbius transformations. He also noted that convergence holds if the radii of the circles $|z - a_k| = r_k$ are sufficiently less than the distances between the centers $|a_k - a_m|$ when $k \neq m$. Beginning from [26, 4] many mathematicians justified the absolute convergence of the Poincaré series under geometrical restrictions to the locations of the circles (see for references [7, 21]). Here, we present such a typical restriction expressed in terms of the separated parameter Δ introduced by Henrici and used by DeLillo *et al.* [9]

$$\Delta = \max_{k \neq m} \frac{r_k + r_m}{|a_k - a_m|} < \frac{1}{(n-1)^{1/4}} \tag{1}$$

for n-connected domain $\mathbb{D}$ bounded by the circles $|z - a_k| = r_k$, $k = 1, 2, \ldots, n$.

In 1916, Myrberg [23, 1] gave examples of absolutely divergent θ_2-series. After this it seemed that the opposite conjectures of Poincaré and Burnside were both wrong. However, it was shown in [18] that θ_2-series converges uniformly for any multiply connected domain $\mathbb{D}$ without any geometrical restriction that corresponds to Burnside's conjecture. Uniform convergence does not directly imply the automorphy relation, i.e. invariance under the Schottky group of transformations, since it is forbidden to change the order of summation without absolute convergence. But this difficulty can be easily overcome by using functional equations. As a result, the Poincaré series satisfies the required automorphy relation and can be written in each fundamental domain with a prescribed summation depending on this domain [18]. The study [18] is based on the solution to a Riemann-Hilbert problem. First, the Riemann-Hilbert problem is written as an $\mathbb{R}$-linear problem which is stated as a conjugation problem between functions analytic in the disks $\mathbb{D}_k = \{z \in \mathbb{C} \colon |z - a_k| < r_k\}$, $k = 1, 2, \ldots, n$, and in $\mathbb{D}$. Further, the latter problem is reduced to a system of functional equations (without integral terms) with respect to the functions analytic in $|z - a_k| < r_k$. The method of successive approximations is justified for this system in a functional space in which convergence is uniform. Straight forward calculations of the successive approximations yields a Poincaré type series (see for instance equation (33)).

General results concerning applications of the boundary value problems to conformal mappings were discussed by Gakhov [11], by Mikhlin [17], by Efendiev and Wendland [12], by Wegert [28]. Recently, an integral equation was constructed by Nasser [25] to calculate the mapping functions from multiply connected domains onto various canonical slit domains. The integral equation was derived by formulating the conformal mapping problem as a Riemann-Hilbert problem. The charge simulation method was used by Amano *et al.* [2] for numerical conformal mappings. Wegmann [29] discussed a numerical method to solve Riemann-Hilbert problems with negative winding numbers.

As noted in the beginning, many important functions can be explicitly constructed as solutions of the special Riemann-Hilbert problems. This method was applied to harmonic measures, Green's function, the Schwarz operator [21] and the Bergmann function [10]. The above results concern explicit constructions of the objects for an arbitrary circular multiply connected domain. In this article, we follow the general idea to construct the conformal mapping of a multiply connected circular domain onto a complex plane with circular slits as a solution of the corresponding Riemann-Hilbert boundary value problem. The connection between the corresponding Riemann-Hilbert problem and the conformal mapping was established in [9]. The final result obtained in this paper generalizes [9] where the restriction on the geometry (1) was used.

2. Schottky group

Consider mutually disjointed disks $\mathbb{D}_k$, $k = 1, 2, \ldots, n$, in the complex plane $\mathbb{C}$ and the multiply connected circular domain $\mathbb{D} = \widehat{\mathbb{C}} \setminus \bigcup_{k=1}^{n} (\mathbb{D}_k \cup L_k)$, the complement of all the closed disks to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Each circle $L_k = \{t \in \mathbb{C} \colon |t - a_k| = r_k\}$ leaves $\mathbb{D}$ to the left. Consider the inversion with respect to the circle L_k,

$$z^*_{(k)} = \frac{{r_k}^2}{\overline{z - a_k}} + a_k.$$

It is known that if a function $\Phi(z)$ is analytic in the disk $|z - a_k| < r_k$ and continuous in its closure, $\overline{\Phi(z^*_{(k)})}$ is analytic in $|z - a_k| > r_k$ and continuous in $|z - a_k| \geq r_k$.

Introduce the composition of successive inversions with respect to the circles $L_{k_1}, L_{k_2}, \ldots, L_{k_p}$

$$z^*_{(k_p k_{p-1} \cdots k_1)} := \left(z^*_{(k_{p-1} \cdots k_1)} \right)^*_{(k_p)}. \tag{2}$$

In the sequence $k_1, k_2, \ldots, k_p$ no two neighboring numbers are equal. The number p is called the level of the mapping. When p is even, these are Möbius transformations. If p is odd, we have anti-Möbius transformations, i.e. Möbius transformations in $\overline{z}$. Thus, these mappings can be written in the form

$$\begin{aligned} \gamma_j(z) &= \frac{e_j z + b_j}{c_j z + d_j}, \qquad \text{for } p \in 2\mathbb{Z}, \\ \gamma_j(\overline{z}) &= \frac{e_j \overline{z} + b_j}{c_j \overline{z} + d_j}, \qquad \text{for } p \in 2\mathbb{Z} + 1, \end{aligned} \tag{3}$$

where $e_jd_j - b_jc_j = 1$, $j = 0, 1, 2, \ldots$. Here

$$\begin{aligned}
\gamma_0(z) &:= z && \text{(identical mapping with the level } p=0\text{)},\\
\gamma_1(\overline{z}) &:= z^*_{(1)}, \ldots, \gamma_n(\overline{z}) := z^*_{(n)} && \text{(} n \text{ simple inversions, } p=1\text{)},\\
\gamma_{n+1}(z) &:= z^*_{(12)}, &&\\
\gamma_{n+2}(z) &:= z^*_{(13)}, &&\\
&\vdots &&\\
\gamma_{n^2}(z) &:= z^*_{(n,n-1)}, && \text{(} n^2 - n \text{ double inversions, } p=2\text{)},\\
\gamma_{n^2+1}(\overline{z}) &:= z^*_{(121)}, &&\\
&\vdots &&
\end{aligned}$$

and so on. The set of the subscripts j of γ_j is ordered in such a way that the level p is increasing. The functions (3) generate a Schottky group $\mathcal{K}$. Thus, each element of $\mathcal{K}$ is presented in the form of a composition of inversions (2) or in the form of linearly ordered functions (3). Let $\mathcal{K}_m$ be such a subset of $\mathcal{K}$ such that the last inversion of each element of $\mathcal{K}_m$ is different from $z^*_{(m)}$, i.e. $\mathcal{K}_m = \{z^*_{(k_pk_{p-1}\cdots k_1)} \colon k_p \neq m\}$. The set $\mathcal{K}'_m = \{z^*_{(k_pk_{p-1}\cdots k_1)} \colon k_1 \neq m\}$ is introduced similarly. All elements γ_j of the even levels generate a subgroup $\mathcal{E}$ of the group $\mathcal{K}$. The set of the elements γ_j of odd level $\mathcal{K}\backslash\mathcal{E}$ is denoted by $\mathcal{O}$. Introduce the notation $\mathcal{E}_m = \mathcal{E} \cap \mathcal{K}_m$, $\mathcal{O}_m = \mathcal{O} \cap \mathcal{K}_m$ and $\mathcal{E}'_m = \mathcal{E} \cap \mathcal{K}'_m$, $\mathcal{O}'_m = \mathcal{O} \cap \mathcal{K}'_m$.

We now proceed to discuss properties of the Möbius and anti-Möbius transformations. Let

$$\gamma(z) = \frac{ez+b}{cz+d}$$

be a Möbius transformation. The following formula can be obtained by straight forward calculations

$$\frac{z-\gamma^{-1}(\tau)}{w-\gamma^{-1}(\tau)} = \frac{\tau-\gamma(z)}{\tau-\gamma(w)} \cdot \frac{cz+d}{cw+d}, \tag{4}$$

where

$$\gamma^{-1}(\tau) = \frac{-d\tau+b}{c\tau-e}$$

is the inverse transformation.

Let us fix an inversion $\zeta^*_{(m)}$. Consider the transformation

$$\gamma_j(z) = (\gamma_t^{-1}(\overline{z}))^*_{(m)}$$

from $\mathcal{E}$, where $\gamma_t \in \mathcal{O}'_m$. It follows from (4) with $\gamma = \gamma_j$ and $\tau = \zeta$ that

$$\frac{z-\gamma_j^{-1}(\zeta)}{w-\gamma_j^{-1}(\zeta)} = \frac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)} \cdot \frac{c_jz+d_j}{c_jw+d_j}. \tag{5}$$

The formula

$$\gamma_t(\overline{z}) = \gamma_j^{-1}(\overline{z}^*_{(m)})$$

implies the following two relations

$$\gamma_j^{-1}(\zeta) = \gamma_t(\overline{\zeta^*_{(m)}}), \qquad \gamma_t(\overline{a_m}) = \gamma_j^{-1}(\infty) = -\frac{d_j}{c_j}. \tag{6}$$

Substitution of (6) into (5) yields

$$\frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} = \frac{z - \gamma_t(\overline{\zeta^*_{(m)}})}{w - \gamma_t(\overline{\zeta^*_{(m)}})} \cdot \frac{w - \gamma_t(\overline{a_m})}{z - \gamma_t(\overline{a_m})}. \tag{7}$$

A formula similar to (4) holds for an anti-Möbius transformation

$$\gamma(\overline{z}) = \frac{e\overline{z} + b}{c\overline{z} + d}$$

namely

$$\frac{w - \gamma^{-1}(\overline{\tau})}{z - \gamma^{-1}(\overline{\tau})} = \overline{\left(\frac{\tau - \gamma(\overline{w})}{\tau - \gamma(\overline{z})}\right)} \cdot \frac{\overline{c}w + \overline{d}}{\overline{c}z + \overline{d}}, \tag{8}$$

where

$$\gamma^{-1}(\overline{z}) = \frac{-\overline{dz} + \overline{b}}{\overline{cz} - \overline{e}}.$$

Consider the transformation

$$\gamma_s(z) = \gamma_j^{-1}(\overline{z^*_{(m)}}),$$

where $\gamma_j \in \mathcal{O}$ and $\gamma_s \in \mathcal{E}'_m$. It follows from (8) with $\gamma = \gamma_j$ and $\tau = \zeta$ that

$$\frac{w - \gamma_j^{-1}(\overline{\zeta})}{z - \gamma_j^{-1}(\overline{\zeta})} = \overline{\left(\frac{\zeta - \gamma_j(\overline{w})}{\zeta - \gamma_j(\overline{z})}\right)} \cdot \frac{\overline{c_j}w + \overline{d_j}}{\overline{c_j}z + \overline{d_j}}. \tag{9}$$

The relation

$$\gamma_j^{-1}(\overline{\zeta}) = \gamma_s(\overline{\zeta^*_{(m)}})$$

implies that

$$\gamma_s(a_m) = \gamma_j^{-1}(\overline{\infty}) = -\frac{\overline{d_j}}{\overline{c_j}}. \tag{10}$$

Substitution of (10) into (9) yields

$$\overline{\left(\frac{\zeta - \gamma_j(\overline{w})}{\zeta - \gamma_j(\overline{z})}\right)} = \frac{w - \gamma_s(\zeta^*_{(m)})}{z - \gamma_s(\zeta^*_{(m)})} \cdot \frac{z - \gamma_s(a_m)}{w - \gamma_s(a_m)}. \tag{11}$$

3. Conformal mapping onto the complex plane with circular slits

Let ζ be a fixed point of $\mathbb{D}\backslash\{\infty\}$. Introduce a conformal mapping $f(z)$ of $\mathbb{D}$ onto a circular slit domain bounded by arcs lying on concentric circles. The mapping is normalized in such a way that $f(\zeta)=0$ and near infinity

$$f(z)=\alpha z+f_0+\frac{f_1}{z}+\frac{f_2}{z^2}+\cdots . \tag{12}$$

Such a conformal mapping $f(z)$ is unique [3, p. 212] up to a multiplier $\alpha\neq 0$. The function

$$S(z)=\frac{f'(z)}{f(z)} \tag{13}$$

is analytic and single-valued in $\mathbb{D}$ except at a fixed point $\zeta\in\mathbb{D}$, where the principal part of $S(z)$ is $(z-\zeta)^{-1}$. According to [9], $S(z)$ satisfies the boundary condition

$$\operatorname{Im}[(t-a_k)S(t)]=0, \qquad |t-a_k|=r_k,\ k=1,2,\ldots,n. \tag{14}$$

The function $S(z)$ plays the main role in the construction of the conformal mapping $f(z)$. DeLillo *et al.* [9] call this function the singularity function $S(z)$. It follows from (12) and (13) that

$$S(z)=\frac{1}{z}+\frac{s_2}{z^2}+\frac{s_3}{z^3}+\cdots, \qquad \text{as } z\to\infty. \tag{15}$$

The function $f(z)$ can be expressed by $S(z)$

$$f(z)=C_0(z-\zeta)\exp[\Omega(z)), \tag{16}$$

where

$$\Omega(z)=\int_w^z\left(S(t)-\frac{1}{t-\zeta}\right)dt, \tag{17}$$

w is a fixed point of $\mathbb{D}$ not equal to ζ or to infinity, C_0 is a constant. DeLillo *et al.* [9] give exact formulae for $S(z)$ and for $f(z)$ in the form of absolutely convergent series and absolutely convergent infinite product [9, (3.36) and (3.41)]

$$f(z)=C(z-\zeta)\prod_{m=1}^{n}\prod_{\gamma_e\in\mathcal{E}'_m}^{\infty}\frac{z-\gamma_e(a_m)}{z-\gamma_e(\zeta^*_{(m)})}\prod_{\gamma_o\in\mathcal{O}'_m}^{\infty}\frac{z-\gamma_o(\overline{\zeta^*_{(m)}})}{z-\gamma_o(\overline{a_m})}, \tag{18}$$

where C is a constant. The infinite product (18) converges absolutely when the circles bounding $\mathbb{D}$ satisfy the separation condition (1).

Let G be a domain on the extended complex plane. Introduce the Banach space $\mathcal{C}(\partial G)$ of functions continuous on ∂G with the norm $\|F\|=\max_{\partial G}|F(t)|$. Let us consider a closed subspace $\mathcal{C}_{\mathcal{A}}(G)$ of $\mathcal{C}(\partial G)$ consisting of functions analytically continued into G. The Maximum Principle implies that convergence in the space $\mathcal{C}_{\mathcal{A}}(G)$ is equivalent to uniform convergence in the closure of G.

We now proceed to construct $f(z)$ without any geometrical restriction on $\mathbb{D}$ using the following Riemann-Hilbert problem

$$\mathrm{Im}\left((t-a_k)\left(\psi_0(t)+\frac{1}{t-\zeta}\right)\right)=0, \qquad |t-a_k|=r_k,\ k=1,2,\ldots,n, \tag{19}$$

with respect to $\psi_0 \in \mathcal{C}_{\mathcal{A}}(\mathbb{D})$. The function $\psi(z)=\psi_0(z)+(z-\zeta)^{-1}$ has similar properties to $S(z)$, i.e. $\psi(z)$ analytic in $\mathbb{D}$ and continuous in $\mathbb{D}\cup\partial\mathbb{D}$ except $\zeta\in\mathbb{D}$, where its principal part is $(z-\zeta)^{-1}$. Moreover, $\psi(\infty)=0$, but the more precise behavior (15) for $\psi(z)$ at infinity is not taken into account yet. The problem (19) can be written in the form

$$\mathrm{Im}((t-a_k)\psi(t))=0, \qquad |t-a_k|=r_k,\ k=1,2,\ldots,n. \tag{20}$$

It follows from (14) that $S(z)$ is one of the solutions of the problem (19).

The winding number of problem (19) (index in other terminology) is equal to

$$\kappa=\sum_{k=1}^{n}\mathrm{wind}_{L_k}\,\overline{(t-a_k)}=n, \tag{21}$$

since the winding number of the function $\overline{(t-a_k)}=r^2(t-a_k)^{-1}$ along L_k is equal to 1, the difference of zeros and poles of $r^2(t-a_k)^{-1}$ in the interior of L_k, i.e. in $|z-a_k|>r_k$ (see [11]). Recall that L_k leaves $\mathbb{D}$ on the left.

Let ℓ be the number of linearly independent solutions (over the field of real numbers) of the homogeneous problem $\mathrm{Im}((t-a_k)\psi_0(t))=0$, $t\in\partial\mathbb{D}$ corresponding to (19), p the number of the linearly independent solvability conditions of the inhomogeneous problem $\mathrm{Im}((t-a_k)\psi_0(t))=g(t)$, $t\in\partial\mathbb{D}$. The relation $\ell-p=2\kappa-n+2$ was shown in [11, (36.50) with $m=n-1$ in our notation], [21] in the class of functions not necessarily equal to zero at infinity. In our case $\psi(z)$ must vanish at infinity, therefore this relation becomes $\ell-p=2\kappa-n$. Moreover, it was established in [11] that in the case $\kappa>n-2$ the inhomogeneous problem always has a solution, hence $p=0$. Taking into account (21) we ultimately obtain that $\ell=n$ and $p=0$. Therefore, the problem (20) as an inhomogeneous Riemann-Hilbert problem always has solutions. The general solution amounts to a partial solution of the inhomogeneous problem and the linear combination of n solutions of the homogeneous problem. The same result follows from Nasser [24, Thm. 4(b)].

In order to solve problem (20) rewrite it in the form of the $\mathbb{R}$-linear problem

$$\begin{aligned}(t-a_k)\psi(t)&=(t-a_k)\psi_k(t)\\ &\quad+\overline{(t-a_k)}\,\overline{\psi_k(t)}+\beta_k, \qquad |t-a_k|=r_k,\ k=1,\ldots,n,\end{aligned} \tag{22}$$

where $\psi_k\in\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ and β_k are undetermined real constants. The problems (20) and (22) are equivalent in the following sense.

Lemma 1.

(i) *If* $\psi(z)$ *and* $\psi_k(z)$ *are solutions of* (22)*, then* $\psi(z)$ *satisfy* (20)*.*
(ii) *If* $\psi(z)$ *is a solution of* (20)*, there exist such functions* $\psi_k(z) \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ *and real constants* β_k *that for each* $k = 1, \ldots, n$ *the* $\mathbb{R}$*-linear condition* (22) *is fulfilled.*

Proof. The proof of the first assertion is evident. It is sufficient to take the imaginary part of (22).

Conversely, let $\psi(z)$ satisfy (20). The function

$$\Psi_k(z) = \frac{\beta_k}{2} + (z - a_k)\,\psi_k(z)$$

can be uniquely determined from the simple Schwarz problem for the disk $\mathbb{D}_k$ [11, 21]

$$2\,\mathrm{Re}\,\Psi_k(t) = \mathrm{Re}(t - a_k)\psi(t), \qquad |t - a_k| = r_k. \tag{23}$$

The latter problem has a unique solution, since $\mathrm{Im}\,\Psi_k(a_k) = 0$. Therefore, the function $\psi_k(z)$ and the constant β_k are uniquely determined in terms of $\psi(z)$ for each $k = 1, \ldots, n$. ■

We now proceed to solve the $\mathbb{R}$-linear problem (22) written in the form

$$\psi(t) = \psi_k(t) + \left(\frac{r_k}{t - a_k}\right)^2 \overline{\psi_k(t)} + \frac{\beta_k}{t - a_k}, \qquad |t - a_k| = r_k,\ k = 1, \ldots, n. \tag{24}$$

The $\mathbb{R}$-linear problem (24) is reduced to functional equations. Following [18, 21] introduce the function

$$\Phi(z) := \begin{cases} \psi_k(z) - \sum\limits_{m \neq k} \left(\dfrac{r_m}{z - a_m}\right)^2 \overline{\psi_m(z^*_{(m)})} - \sum\limits_{m \neq k} \dfrac{\beta_m}{z - a_m}, & |z - a_k| \leq r_k, \\ & k = 1, 2, \ldots, n, \\ \psi(z) - \sum\limits_{m=1}^{n} \left(\dfrac{r_m}{z - a_m}\right)^2 \overline{\psi_m(z^*_{(m)})} - \sum\limits_{m=1}^{n} \dfrac{\beta_m}{z - a_m}, & z \in \mathbb{D}, \end{cases}$$

analytic in the domains $\mathbb{D}_k$, $k = 1, 2, \ldots, n$, and $\mathbb{D}\backslash\{\zeta\}$. Calculate the jump across the circle L_k

$$\Delta_k := \Phi^+(t) - \Phi^-(t), \qquad |t - a_k| = r_k,$$

where

$$\Phi^+(t) := \lim_{z \to t,\, z \in \mathbb{D}} \Phi(z),$$
$$\Phi^-(t) := \lim_{z \to t,\, z \in \mathbb{D}_k} \Phi(z).$$

Using (24) we get $\Delta_k = 0$. It follows from the Analytic Continuation Principle that $\Phi(z)$ is analytic in the extended complex plane except at the point ζ where

$\Phi(z) \sim (z-\zeta)^{-1}$. Moreover, $\psi(\infty) = 0$ yields $\Phi(\infty) = 0$. Then Liouville's Theorem implies that $\Phi(z) = (z-\zeta)^{-1}$. The definition of $\Phi(z)$ in $|z-a_k| \leq r_k$ yields the following system of functional equations

$$\psi_k(z) = \sum_{m \neq k} \left(\frac{r_m}{z-a_m} \right)^2 \overline{\psi_m(z^*_{(m)})} + h_k(z), \tag{25}$$

where the function

$$h_k(z) = \sum_{m \neq k} \frac{\beta_m}{z-a_m} + \frac{1}{z-\zeta}, \qquad |z-a_k| \leq r_k,\ k = 1, \dots, n \tag{26}$$

belongs to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$. It follows from the definition of $\Phi(z)$ in $\mathbb{D}$ that the general solution of the Riemann-Hilbert problem (20) is constructed via $\psi_k(z)$

$$\psi(z) = \sum_{m=1}^{n} \left(\frac{r_m}{z-a_m} \right)^2 \overline{\psi_m(z^*_{(m)})} + \frac{1}{z-\zeta} + \sum_{m=1}^{n} \frac{\beta_m}{z-a_m}, \qquad z \in \mathbb{D} \cup \partial\mathbb{D}. \tag{27}$$

The function $\psi(z)$ analytic in $\mathbb{D}$ except at the point ζ where its principal part is $(z-\zeta)^{-1}$.

Lemma 2 ([21, Lem. 4.8, p. 167])**.** *The system* (25) *has a unique solution in* $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$*,* $k = 1, 2, \dots, n$*. This solution can be found by the method of successive approximations.*

Let $\psi_k(z)$ be a solution to the system of functional equations (25). Let $w \in \mathbb{D}$, $w \neq \zeta, \infty$, be the fixed point introduced in (17). Introduce the functions

$$\varphi_m(z) = \int_{w^*_{(m)}}^{z} \psi_m(t)\, dt + \varphi_m(w^*_{(m)}), \quad m = 1, 2, \dots, n, \tag{28}$$

and

$$\omega(z) = -\sum_{m=1}^{n} \left(\overline{\varphi_m(z^*_{(m)})} - \overline{\varphi_m(w^*_{(m)})} \right) + \sum_{m=1}^{n} \beta_m \ln \frac{z-a_m}{w-a_m}. \tag{29}$$

Here, the following relation from [21] is used

$$\frac{d}{dz} \left(\overline{\varphi_m(z^*_{(m)})} \right) = -\left(\frac{r_k}{z-a_k} \right)^2 \overline{\frac{d\varphi_m}{dz}(z^*_{(m)})}, \qquad |z-a_k| > r_k. \tag{30}$$

The functions $\omega(z)$ and $\varphi_m(z)$ belong to $\mathcal{C}_{\mathcal{A}}(\mathbb{D})$ and to $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_m)$, respectively. One can see from (28) that the function $\varphi_m(z)$ is determined by $\psi_m(z)$ up to an additive constant which vanishes in (29). The function $\omega(z)$ vanishes at $z = w$.

Integrate each functional equation (25). Application of (28) yields functional equations with respect to $\varphi_k \in \mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$

$$\begin{aligned}(31) \qquad \varphi_k(z) = &-\sum_{m\neq k}\left(\overline{\varphi_m(z^*_{(m)})} - \overline{\varphi_m(w^*_{(m)})}\right) + \ln(z-\zeta)\\ &+\sum_{m\neq k}\beta_m \ln(z-a_m) + c_k, \qquad |z-a_k| \le r_k,\ k=1,\dots,n,\end{aligned}$$

where c_k are undetermined constants. A single valued branch of the logarithm $\ln Z$ is fixed in such a way that all cuts of the logarithms in the right part of (31) lie in the corresponding domains $\mathbb{D} \cup \mathbb{D}_m \cup \partial\mathbb{D}_m$ and $\ln 1 = 0$. The following result in another form was obtained in [21].

Lemma 3. *The system* (31) *with fixed* c_k *has a unique solution in* $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$, $k=1,\dots,n$. *This solution can be found by the method of successive approximations.*

Proof. We give a short proof of the lemma so that the remainder of the paper is clear. The proof follows from Lemma 2, since (31) is the result of the integral operator

$$(32) \qquad F \mapsto \int_{w^*_{(k)}}^{z} F(t)\,dt$$

applied to (25). Convergence in $\mathcal{C}_{\mathcal{A}}(\mathbb{D}_k)$ means uniform convergence in $\mathbb{D}_k \cup \partial\mathbb{D}_k$. Therefore, the integral operator (32) can be applied term by term to the successive approximations for (25). This yields the uniformly convergent successive approximations for (31). ■

Application of the method of successive approximations to (31) yields the uniformly convergent series

$$\begin{aligned}(33) \qquad \varphi_k(z) = &\underbrace{c_k + \ln(z-\zeta) + \sum_{k_1\neq k}\beta_{k_1}\ln(z-a_{k_1})}_{\text{0th approx.}}\\ &\underbrace{-\sum_{k_1\neq k}\ln\frac{\overline{\zeta - z^*_{(k_1)}}}{\overline{\zeta - w^*_{(k_1)}}} - \sum_{k_1\neq k}\sum_{k_2\neq k_1}\beta_{k_2}\ln\frac{\overline{a_{k_2} - z^*_{(k_1)}}}{\overline{a_{k_2} - w^*_{(k_1)}}}}_{\text{1st approx.}}\end{aligned}$$

$$+\sum_{k_1\neq k}\sum_{k_2\neq k_1}\ln\frac{\zeta - z^*_{(k_2k_1)}}{\zeta - w^*_{(k_2k_1)}}$$

$$\underbrace{+\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}\beta_{k_3}\ln\frac{a_{k_3} - z^*_{(k_2k_1)}}{a_{k_3} - w^*_{(k_2k_1)}}}_{\text{2nd approx.}}$$

$$-\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}\ln\frac{\overline{\zeta - z^*_{(k_3k_2k_1)}}}{\overline{\zeta - w^*_{(k_3k_2k_1)}}}$$

$$\underbrace{-\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}\sum_{k_4\neq k_3}\beta_{k_4}\ln\frac{\overline{a_{k_4} - z^*_{(k_3k_2k_1)}}}{\overline{a_{k_4} - w^*_{(k_3k_2k_1)}}}}_{\text{3d approx.}}$$

$$+\ldots, \qquad |z - a_k| \leq r_k.$$

Here, uniform convergence is understood in such a way that each term of the series coincides with a successive approximation shown in (33). The order of summation into each approximation can be arbitrarily fixed. It is worth noting that an approximation of order p contains the Möbius mapping of the same level p.

The function (29) can be written in the form of the uniformly convergent product

$$\omega(z) = \ln\Bigg(\prod_{k=1}^{n}\left(\frac{a_k - z}{a_k - w}\right)^{\beta_k}\left[\prod_{k=1}^{n}\frac{\overline{\zeta - w^*_{(k)}}}{\overline{\zeta - z^*_{(k)}}}\prod_{k_1\neq k}\left(\frac{\overline{a_{k_1} - w^*_{(k)}}}{\overline{a_{k_1} - z^*_{(k)}}}\right)^{\beta_{k_1}}\right]$$
$$\times\prod_{k=1}^{n}\prod_{k_1\neq k}\left[\frac{\zeta - z^*_{(k_1k)}}{\zeta - w^*_{(k_1k)}}\prod_{k_2\neq k_1}\left(\frac{a_{k_2} - z^*_{(k_1k)}}{a_{k_2} - w^*_{(k_1k)}}\right)^{\beta_{k_2}}\right]$$
$$\times\prod_{k=1}^{n}\prod_{k_1\neq k}\prod_{k_2\neq k_1}\left[\frac{\overline{\zeta - w^*_{(k_2k_1k)}}}{\overline{\zeta - z^*_{(k_2k_1k)}}}\prod_{k_3\neq k_2}\left(\frac{\overline{a_{k_3} - w^*_{(k_2k_1k)}}}{\overline{a_{k_3} - z^*_{(k_2k_1k)}}}\right)^{\beta_{k_3}}\right]\cdots\Bigg). \tag{34}$$

The series (33) can be decomposed onto a sum of $n+1$ (or onto the sum of two) uniformly convergent series in the following way. Functional equations (31) are linear. Therefore, their solution $\varphi_k(z)$ can be presented as the sum

$$\varphi_k(z) = \varphi_k^{(0)}(z) + \sum_{m=1}^{n}\beta_m\varphi_k^{(m)}(z). \tag{35}$$

The function $\varphi_k^{(0)}(z)$ is constructed by application of the successive approximations separately to $\ln(z-\zeta)$; $\varphi_k^{(m)}(z)$ separately to $(1-\delta_{km})\ln(z-a_m)$, where δ_{km} is the Kronecker symbol. The function $\omega(z)$ can be decomposed also like (35). However, it is convenient to use the decomposition into two functions

$$\omega(z) = \omega_0(z) + \phi(z;\beta_1,\beta_2,\dots,\beta_n), \tag{36}$$

where

$$\omega_0(z) = -\sum_{k=1}^{n}\left(\overline{\varphi_k^{(0)}(z_{(k)}^*)} - \overline{\varphi_k^{(0)}(w_{(k)}^*)}\right) \tag{37}$$

and

$$\phi(z;\beta_1,\beta_2,\dots,\beta_n) = -\sum_{k=1}^{n}\left(\overline{\phi_k(z_{(k)}^*)} - \overline{\phi_k(w_{(k)}^*)}\right) + \sum_{k=1}^{n}\beta_k\ln(z-a_k). \tag{38}$$

The functions $\phi_k(z)$ are constructed by application of the successive approximations to $\sum_{m=1}^{n}\beta_m(1-\delta_{km})\ln(z-a_m)$, $k = 1,2,\dots,n$. Uniqueness of the solution of the system (31) yields the same result not depending on the manner of decomposition

$$\phi_k(z) = \sum_{m=1}^{n}\beta_m\varphi_k^{(m)}(z), \qquad k = 1,2,\dots,n. \tag{39}$$

The function $\omega_0(z)$ can be calculated from (34) with all $\beta_k = 0$. It can be also represented in the form

$$\omega_0(z) = \ln\prod_{j=1}^{\infty}\mu_j(z,\zeta), \tag{40}$$

where

$$\mu_j(z,\zeta) = \begin{cases} \dfrac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)} & \text{if } \gamma_j\in\mathcal{E}, \\[2ex] \dfrac{\overline{\zeta-\gamma_j(\overline{w})}}{\overline{\zeta-\gamma_j(\overline{z})}} & \text{if } \gamma_j\in\mathcal{O}. \end{cases} \tag{41}$$

The multipliers $\mu_j(z,\zeta)$ in (40) are arranged in accordance with the increasing level of γ_j. The function $\phi(z;\beta_1,\beta_2,\dots,\beta_n)$ was constructed in [21] in the form

$$\phi(z;\beta_1,\beta_2,\dots,\beta_n) = \sum_{m=1}^{n}\beta_m\omega_m(z), \tag{42}$$

where

$$\omega_m(z) = \ln\Bigg((z-a_m)\left(\prod_{k\neq m}\frac{\overline{a_m - w^*_{(k)}}}{\overline{a_m e - z^*_{(k)}}}\right)\left(\prod_{k_1\neq m}\prod_{k\neq k_1}\frac{a_m - z^*_{(k_1 k)}}{a_m - w^*_{(k_1 k)}}\right) \times \prod_{k_1\neq m}\prod_{k_2\neq k_1}\prod_{k\neq k_2}\left(\frac{\overline{a_m - w^*_{(k_1 k_2 k)}}}{\overline{a_m - z^*_{(k_1 k_2 k)}}}\right)\cdots\Bigg). \tag{43}$$

The latter infinite product can be also written in the form

$$\omega_m(z) = \ln \prod_{\gamma_j\in\mathcal{K}_m}^{\infty} \mu_j(z, a_m). \tag{44}$$

The general solution of the Riemann-Hilbert problem (19) (or (20)) can be written exactly by calculation of the derivatives $\psi_0(z) = \omega'(z)$ by use of formula (34). It depends on n arbitrary real constants β_m. The explicit form of $\psi_0(z)$ is not written here because the function $\omega(z)$ is needed for the conformal mapping. Substitute $\psi_0(z)$ into (16) instead of $\Omega(z)$ which is equivalent to substitution of $\psi(z)$ into (17). Integration in (16) of the uniformly convergent series term by term yields the function

$$F(z;\beta_1,\beta_2,\ldots,\beta_n) = \prod_{j=0}^{\infty}\mu_j(z,\zeta)\cdot\prod_{m=1}^{n}\left[\prod_{\gamma_j\in\mathcal{K}_m}^{\infty}\mu_j(z,a_m)\right]^{\beta_m}. \tag{45}$$

Theorem 4. *The function* (45) *yields the required conformal mapping when all* β_m *vanish*

$$f(z) = \prod_{j=0}^{\infty}\mu_j(z,\zeta). \tag{46}$$

Proof. The function $F(z;\beta_1,\beta_2,\ldots,\beta_n)$ is constructed on the basis of the general solution $\psi(z)$ of the Riemann-Hilbert problem (20) which contains the singularity function (13). Therefore, $F(z;\beta_1,\beta_2,\ldots,\beta_n)$ contains the conformal mapping (16). Hence, in order to prove the theorem it is sufficient to check that all zeros and poles of the function (45) coincide with the zeros and poles of the required conformal mapping if and only if all β_m are equal to zero.

DeLillo *et al.* [9] used the Schwarz reflection principle to prove that all zeros and poles of the conformal mapping $f(z)$ are simple. The location of the zeros and poles of $f(z)$ is explicitly shown in formula (18). Demonstration that the function (46) has the same zeros and poles as (18) is based on formulae (7) and (11). Let $\mu_j(z,\zeta) = (\zeta-\gamma_j(z))/(\zeta-\gamma_j(w))$ be a multiplier of (46) with $\gamma_j \in \mathcal{E}$ (see (41)). Then (7) implies that this multiplier has a simple zero at $z = \gamma_e(\zeta)$, where $\gamma_e(\zeta) = \gamma_t(\overline{\zeta^*_{(m)}})$ belongs to $\mathcal{E}$, and has a simple pole at $z = \gamma_t(\overline{a_m})$, where $\gamma_t \in \mathcal{O}'_m$. Similar arguments can be applied to $\mu_j(z,\zeta)$ with $\gamma_j \in \mathcal{O}$ by use of (11) to show that $\mu_j(z,\zeta)$ has a simple zero at $z = \gamma_s(a_m)$, where $\gamma_s \in \mathcal{E}'_m$ and has a simple pole at $z = \gamma_o(\overline{\zeta})$, where $\gamma_o(\overline{\zeta}) = \gamma_s(\zeta^*_{(m)})$. ■

4. Discussion

It is worth noting that (46) implies the partial formula (18) after application of (7) and (11). The essential difference between (18) and (46) is explicitly seen if the functions $\mu_j(z,\zeta)$ defined by (41) are transformed by (7) or by (11). For instance, the transformed multiplier in (46)

$$\mu_j(z,\zeta) = \frac{z - \gamma_t(\overline{\zeta^*_{(m)}})}{w - \gamma_t(\overline{\zeta^*_{(m)}})} \cdot \frac{w - \gamma_t(\overline{a_m})}{z - \gamma_t(\overline{a_m})} \qquad \text{for } \gamma_j \in \mathcal{E} \tag{47}$$

corresponds to the multiplier in (18)

$$\frac{z - \gamma_t(\overline{\zeta^*_{(m)}})}{z - \gamma_t(\overline{a_m})}. \tag{48}$$

All such multipliers coincide only if $w = \infty$. However, this infinite point is excluded from (17) in order to integrate the uniformly convergent series term by term on a finite arc to construct the functions (45) and (46). So, the point $w = \infty$ is unique for which the products (45) and (46) can be uniformly and absolutely divergent. Uniformly, because an infinite arc of the integration is taken; absolutely, because of the example presented in [1]. This complicated situation can be illustrated by a simple example. Let the almost uniformly convergent series $\sum_{n=1}^{\infty}(n-z)^{-2}$, $z \notin \mathbb{N}$, be integrated term by term

$$\int_w^z \sum_{n=1}^{\infty} \frac{1}{(n-t)^2}\,dt = \sum_{n=1}^{\infty}\left(\frac{1}{n-z} - \frac{1}{n-w}\right).$$

One can see that this series is convergent if and only if $w \neq \infty$. This “unlucky” infinity is sometimes taken as a fixed point in similar investigations by specialists in complex analysis (see for instance Michlin’s study [17] about convergence of Schwarz’s method [21]).

The conformal mapping (46) is constructed as a solution of the Riemann-Hilbert problem by general formula (45) in order to extend the method of functional equations presented in [18, 21]. The main advantage of the method is that formulae (45) and (46) are given without any geometrical restriction on the location of the circles L_k.

It is worth noting that the method can be effectively realized in computations since it is based on the direct iterative scheme when the system of functional equations (31) can be solved by successive approximations. This method can be modified to look for the solution in the form of the series on multipliers of the form ${r_l}^2|a_k - a_m|^{-2}$ following [19]. This is equivalent to the truncated product representation (46). The numerical implementation of the formula (46) is the same as that in [9] when absolute convergence takes place. It is interesting to find an example of the domain $\mathbb{D}$ for which the product (18) uniformly (not absolutely) diverges and simultaneously the product (46) uniformly converges.

Crowdy [7] proposed another general method to construct some important functions of complex analysis. In particular, the conformal mapping discussed was constructed in [8] in terms of the Schottky-Klein prime function. This result is close to [9] because the Schottky-Klein prime function can be presented explicitly by an infinite product for which convergence is known when a separation condition holds. It will be interesting to investigate the explicit representation for the Schottky-Klein prime function in the general case via a Riemann-Hilbert problem. Now, it is not clear how to write an appropriate Riemann-Hilbert problem for the Schottky-Klein prime function as was done by DeLillo *et al.* [9] for the conformal mapping.

The same convergence question arises for the vector-matrix Riemann-Hilbert problem. The proof of uniform convergence presented in [20] is wrong. It is corrected in [22] where absolute convergence of the corresponding series was justified when a separation condition holds. The general case was reduced via functional equations to an infinite system of linear algebraic equations. Application of the method of truncation to this infinite system yields a finite system of linear algebraic equations and auxiliary functional equations which can be solved by the absolutely convergent method of successive approximations. However, such a method does not lead to an analytical form of the final formulae. So, the uniform convergence of the Poincaré vector-matrix series for arbitrary multiply connected domain needs further investigation.

References

1. T. Akaza, Singular sets of some Kleinian groups, *Nagoya Math. J.* **26** (1966), 127–143.
2. K. Amano, D. Okano, H. Ogata and M. Sugihara, Numerical conformal mappings of unbounded multiply-connected domains using the charge simulation method, *Malaysian Math. Sc. Soc. (Second Series)* **26** (2003), 35–51.
3. L. Bieberbach, *Conformal Mapping*, AMS Chelsea Publ., New York, 2000.
4. W. Burnside, On a class of automorphic functions, *Proc. London Math. Soc.* **23** (1891), 49–88.
5. D. Crowdy and H. Kang, Squeeze flow of multiply-connected fluid domains in a Hele-Shaw cell, *J. Nonlinear Sci.* **11** (2001), 279–304.
6. D. Crowdy and J. Marshall, Analytical formulae for the Kirchhoff-Routh path function in multiply connected domains, *Proc. R. Soc.* **A 461** (2005), 2477–2501.
7. D. Crowdy, Geometric function theory: a modern view of a classical subject, *Nonlinearity* **21** no.10 (2008), T205–T219.
8. ———, Conformal mappings between canonical multiply connected domains, *Comput. Methods Function Theory* **6** no.1 (2006), 59–76.
9. T. K. DeLillo, T. A. Driscoll, A. R. Elcrat and J. A. Pfaltzgraff, Radial and circular slit maps of unbounded multiply connected circle domains, *Proc. R. Soc. London* **A 464** (2008), 1719–1737.

10. J. Moonja, V. Mityushev, The Bergman kernel for circular multiply connected domains, *Pacific J. Math.* **233** (2007), 145–157.
11. F. D. Gakhov, *Boundary Value Problems*, Dover Publications Inc., New York, 1990.
12. M. A. Efendiev and W. L. Wendland, Geometrical properties of nonlinear maps and their application; Part II: Nonlinear Riemann-Hilbert problems with closed boundary data for multiply connected domains, *J. Math. Anal. Appl.* **329** (2007), 425–444.
13. S. Giordano, Multipole analysis of a generic system of dielectric cylinders and application to fibrous materials, *J. Electrostatics* **63** (2005), 1–19.
14. A. L. Kalamkarov, I. V. Andrianov and V. V. Danishevskyy, Asymptotic homogenization of composite materials and structures, *Appl. Mech. Rev.* **62** no.3 (2009), 030802, 20 pages.
15. P. A. Krutitskii, The mixed harmonic problem in a bounded cracked domain with Dirichlet condition on cracks, *J. Differential Equations* **198** no.2 (2004), 422–441.
16. ———, The Dirichlet-Neumann harmonic problem in a two–dimensional cracked domain with the Neumann condition on cracks, *Proc. R. Soc. Lond.* **A 460** (2004), 445–462.
17. S. G. Mikhlin, *Integral Equations*, Pergamon Press, New York, 1964.
18. V. V. Mityushev, Convergence of the Poincaré series for classical Schottky groups, *Proc. Amer. Math. Soc.* **126** no.8 (1998), 2399–2406.
19. ———, Transport properties of doubly periodic arrays of circular cylinders and optimal design problems, *Appl. Math. Opt.* **44** (2001), 17–31.
20. ———, Riemann problem on double of multiply connected region, *Ann. Polon. Math.* **77.1** (1997), 1–14.
21. V. V. Mityushev and S. V. Rogosin, *Constructive Methods for Linear and Non-Linear Boundary Value Problems of the Analytic Function. Theory and Applications*, Monographs and Surveys in Pure and Applied Mathematics, Chapman & Hall / CRC, Boca Raton etc., 2000.
22. ———, On the Riemann-Hilbert problem with a piecewise constant matrix, *Z. Anal. Anwend.* **27** (2008), 53–66.
23. P. J. Myrberg, Zur Theorie der Konvergenz der Poincaréschen Reihen, *Ann. Acad. Sci. Fennicae* **A9** no.4 (1916), 1–75.
24. M. M. S. Nasser, The Riemann-Hilbert problem and the generalized Neumann kernel on unbounded multiply connected regions, *The University Researcher (IBB University Journal)* **20** (2009), 47–60.
25. ———, Numerical conformal mapping via a boundary integral equation with the generalized Neumann kernel, *SIAM J. Sci. Comput.* **31** (2009), 1695–1715.
26. H. Poincaré, *Oeuvres*, Gauthier-Villart, Paris, v. 2 1916, v. 4 1950, v. 9 1954.
27. W. J. Prosnak *Computation of Fluid Motions in Multiply Connected Domains*, Braun, Witzwort, 1987.
28. E. Wegert *Nonlinear Boundary Value Problems for Holomorphic Functions and Singular Integral Equations*, Akademie-Verlag, Berlin, 1992.
29. R. Wegmann, Constructive solution of a certain class of Riemann-Hilbert problems on multiply connected circular domains, *J. Comput. Appl. Math.* **130** (2001), 139–161.
30. E. I. Zverovich, Boundary value problems of analytic functions in Hölder classes on Riemann surfaces, *Uspekhi Mat. nauk* **26** no.1 (1971), 113–179 (in Russian).

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