

Conductivity of a two-dimensional composite containing elliptical inclusions

Vladimir Mityushev

Proc. R. Soc. A 2009 **465**, 2991-3010 first published online 8 July 2009
doi: 10.1098/rspa.2009.0219

References

This article cites 24 articles

<http://rspa.royalsocietypublishing.org/content/465/2110/2991.full.html#ref-list-1>

Subject collections

Articles on similar topics can be found in the following collections

[mathematical physics](#) (87 articles)

Email alerting service

Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click [here](#)

To subscribe to *Proc. R. Soc. A* go to: <http://rspa.royalsocietypublishing.org/subscriptions>

Conductivity of a two-dimensional composite containing elliptical inclusions

BY VLADIMIR MITYUSHEV*

*Department of Mathematics, Pedagogical University, ul. Podcharznych 2,
 Krakow 30-084 Poland*

We develop a method of functional equations to derive analytical approximate formulae for the effective conductivity tensor of the two-dimensional composites with elliptical inclusions. The sizes, the locations and the orientations of the ellipses can be arbitrary. The analytical formulae contain all the above geometrical parameters in symbolic form.

Keywords: transport properties; effective conductivity; elliptical inclusions

1. Introduction

Analysis concerning the transport properties of inhomogeneous materials is of fundamental theoretical interest. It is also important in engineering, for instance in optimal design problems. Analytical formulae for the macroscopic properties with physical and geometrical parameters in symbolic form are useful to predict the behaviour of composites. Using the terminology of electric or thermal conductivity, we study the transport properties of a two-dimensional, two-phase composite with parallel elliptical cylinders of conductivity λ embedded in a host material of the normalized unit conductivity.

The effective conductivity of random conducting materials has been studied since the nineteenth century by Maxwell and Rayleigh. Most of the work in literature is concerned with dilute composites with spherical and ellipsoidal inclusions (circular and elliptical inclusions in two-dimensional composites), because the corresponding boundary value problem for one inclusion in the whole space can be solved in closed form. The attention of this study is on two-dimensional composites for which few analytical formulae are known.

Let ν be the area fraction of the inclusions, and

$$\rho = \frac{\lambda - 1}{\lambda + 1} \quad (1.1)$$

denotes the contrast parameter introduced by Bergman (1978). The semi-axes of the ellipses are denoted by $r(1 + \alpha)$ and $r(1 - \alpha)$, where $0 < \alpha < 1$ and $r > 0$. If all the ellipses are aligned with the coordinate axes, the effective conductivity tensor

*vmityushev@gmail.com

is diagonal. Its diagonal components were estimated by Galeener (1971*a–c*):

$$\lambda_x \approx 1 + \frac{2\rho\nu}{1 - \rho(\nu + \alpha)} \quad \text{and} \quad \lambda_y \approx 1 + \frac{2\rho\nu}{1 - \rho(\nu - \alpha)}. \quad (1.2)$$

Cohen *et al.* (1973) discussed what they viewed as inconsistencies arising from formulae (1.2) and derived formulae based on an elliptical Lorentz cavity having the same depolarization factors as the inclusions:

$$\lambda_x \approx 1 + \frac{2\rho\nu}{1 - \rho(\alpha + \nu(1 - \alpha))} \quad \text{and} \quad \lambda_y \approx 1 + \frac{2\rho\nu}{1 + \rho(\alpha - \nu(1 + \alpha))}. \quad (1.3)$$

The scalar macroscopic conductivity of randomly distributed ellipses were calculated by Hetherington & Thorpe (1992) and by Zimmerman (1996):

$$\lambda_e = 1 + \frac{2\rho\nu}{1 - \alpha^2\rho^2} + O(\nu^2). \quad (1.4)$$

Having applied the differential method, Zimmerman (1996) deduced another approximate formula

$$\frac{1}{1 - \nu} \approx \lambda_e^{(1-\alpha^2)/2} \left| \frac{\lambda - 1}{\lambda - \lambda_e} \right| \left| \frac{\lambda + \lambda_e}{\lambda + 1} \right|^{\alpha^2}. \quad (1.5)$$

The following formula was obtained by Hetherington & Thorpe (1992) in the case of holes whose shape is an n -sided regular polygon

$$\lambda_e = 1 - \frac{\tan(\pi/n)\Gamma^4(1/n)}{2\pi n\Gamma^2(2/n)} + O(\nu^2), \quad (1.6)$$

where Γ is a gamma function. Garboczi & Douglas (1996) investigated the influence of the various shapes of inclusions on the effective conductivity up to $O(\nu^2)$. Ammari *et al.* (2005) expressed the effective conductivity tensor Λ via the dipole matrix M

$$\Lambda \approx I - \frac{\nu}{\pi} M \left(I + \frac{\nu}{2\pi} M \right)^{-1}, \quad (1.7)$$

where I is the identity matrix. A method of integral equations was applied to compute the matrix M . Movchan *et al.* (2002, 2003) explicitly calculated M for the shapes described by algebraic equations. Crowdy (2008) applied the Schottky–Klein prime function to effectively compute the local field of finite sets of discs. Having used the dipole matrix method, Rylko (2000) deduced formulae for the principal conductivities of a dilute composite with curvilinear inclusions unidirectionally located on the plane

$$\lambda_1 = \frac{1 + \rho\nu(1 + \varepsilon\rho|c_2|)}{1 - \rho\nu(1 + \varepsilon\rho|c_2|)} + O(\nu^2) \quad \text{and} \quad \lambda_2 = \frac{1 + \rho\nu(1 - \varepsilon\rho|c_2|)}{1 - \rho\nu(1 - \varepsilon\rho|c_2|)} + O(\nu^2), \quad (1.8)$$

where the shape of the inclusion is defined by the function $r = 1 + \varepsilon f(\theta)$ in polar coordinates (r, θ) , ε is a small positive parameter, and

$$c_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) e^{2i\theta} d\theta.$$

General methods to deduce formulae (1.2)–(1.5) and discussion of other approaches were presented by Milton (2002).

The summarized works are based mainly on the solution to the boundary value problems with one inclusion in the plane. It should be emphasized that the above-discussed formulae are valid only to the first order of ν . In general, to evaluate higher-order terms, we need to discuss boundary value problems with an infinite number of inclusions and to investigate at least the first-order interactions between inclusions. Such problems in a class of doubly periodic functions for composites with circular inclusions were systematically studied by Berlyand & Mityushev (2001, 2005), Mityushev (2001, 2005) and Mityushev *et al.* (2008). In particular, an exact formula for the effective conductivity of the square array of cylinders was constructed. Approximate analytical formulae for the effective conductivity tensor for any distribution of the discs were deduced.

The problem of determining the conductivity of randomly distributed ellipses has not yet been dealt with rigorously for non-dilute composites. As has already been noted, Maxwell's approach is based on a single inclusion and can produce formulae valid only to the first order of ν . Such approximations take into account the shape of inclusions, but not the geometry of their location. To the best of our knowledge, there are only four works devoted to the rigorous study of composites with many elliptical inclusions with the precision at least $O(\nu^2)$. McPhedran & Nicorovici (1997) and Nicorovici & McPhedran (1996) applied Rayleigh's method to a rectangular array of elliptical cylinders and reduced the problem to an infinite system of linear algebraic equations. Analytical solution to a truncated system for sufficiently small α yields

$$\lambda^x \approx 1 + \frac{2\rho\nu}{1 - \rho(\alpha + (S_2/\pi)\nu)} \quad \text{and} \quad \lambda^y \approx 1 + \frac{2\rho\nu}{1 - \rho(\alpha - ((2\pi - S_2)/\pi)\nu)}, \quad (1.9)$$

where S_2 is a lattice sum. More precise formulae for not small α were also deduced. The later ones differ from equation (1.9) by a coefficient on ν in the denominator. Formulae (1.9) and others presented by McPhedran & Nicorovici (1997) and Nicorovici & McPhedran (1996) are rigorous, because the applied Rayleigh's method takes into account both the shape of the ellipses and the geometry of the array, by means of the lattice sum.

Yardley *et al.* (1999, 2001) calculated the effective conductivity for a structure consisting of a finite number of layers, or stack, of ellipses by Rayleigh's method. It is interesting to extend the results by McPhedran & Nicorovici (1997), Nicorovici & McPhedran (1996) and Yardley *et al.* (1999, 2001) to random media by using the method of functional equations that effectively works for circular inclusions.

In this paper, the first step has been done in this direction. We propose an iterative method to compute the effective conductivity tensor Λ for an arbitrary distribution of non-overlapping ellipses. It is based on symbolic computations and yields analytical formulae for the local field and for Λ in the form of a series in ν .

Table 1. Nomenclature.

X, Y	local coordinates for each ellipse
x, y	global coordinates
$Z = X + iY, z = x + iy, w$	complex coordinates
D_m	elliptical inclusions with the number m
D	domain occupied by host material
$r_m(1 + \alpha), r_m(1 - \alpha)$	semi-axes of the ellipse ∂D_m
θ_m	angle between the major axis of ∂D_m and the x -axis
$s_m = r_m e^{i\theta_m}$	complex parameter of ∂D_m
$a_m = x_m + iy_m$	centre of the ellipse D_m
λ	normalized conductivity of inclusions
ρ	contrast parameter
n	total number of inclusions
N	number of inclusions per periodicity cell
ν	concentration of inclusions
$\Lambda = \begin{pmatrix} \lambda^x & \lambda^{xy} \\ \lambda^{xy} & \lambda^y \end{pmatrix}$	effective conductivity tensor
λ_e	effective conductivity of macroscopically isotropic composite
u_m, φ_m	potentials in D_m
u, φ	potentials in D
$\phi_m, \psi_m = \phi'_m, \Phi_m$	complex potentials in $\sqrt{\alpha} \leq w \leq 1$
t	complex variable on the ellipses
τ	complex variable on the circles $ \tau = \sqrt{\alpha}$ and $ \tau = 1$
$\ell = \max_{k \neq m} \sqrt{r_k r_m} a_k - a_m ^{-1}$	characteristic ratio
$S_2(n)$	finite sum (equation (4.25))
S_2	infinite sum (equation (4.30))
S_2	lattice sum of second order

The final formulae (4.34)–(4.37) for Λ are derived up to $O(\nu^3)$. One can proceed the iterations and obtain higher-order formulae. However, it requires advanced symbolic computations.

This paper is organized as follows. A list of the nomenclature is given in table 1. The problem is presented in §2, and the \mathbb{R} -problem that governs the local complex potential is derived in §2*a*. The problem is reduced to integral equations in §2*c*. The main part of the paper, solution to the integral equations, is presented in §3. Applications to the calculation of the effective conductivity tensor are presented in §4. New analytical formulae up to $O(\nu^3)$ are summarized in §4*c–f*. They are universal for any distribution of the centres of inclusions, of their sizes and of orientations. This analytical dependence of Λ on locations of elliptical inclusions was not known before. Some concluding remarks are given in §5.

2. Statement of the problem and reduction to integral equations

(a) Geometry

Non-overlapping elliptical inclusions D_m ($m = 1, 2, \dots, n$) are embedded in the plane (x, y) . It is convenient to put the semi-axes equal to $r_m(1 + \alpha)$ and $r_m(1 - \alpha)$, respectively. The positive parameter r_m characterizes the size of the

inclusion, and α the shape of the ellipse ($0 < \alpha < 1$). The limit case when α is equal to 0 can be treated in the final formulae by calculating the corresponding limits. The limit case $\alpha = 1$ requires a separate investigation that is not presented in this paper. Let an inclusion D_m be centred at (x_m, y_m) and the angle between the major semi-axis of the ellipse and the x -axis is equal to θ_m . Introduce the local coordinates (X, Y) for a fixed inclusion D_m as follows:

$$\text{and } \left. \begin{aligned} X &= \frac{1}{r_m} [(x - x_m) \cos \theta_m + (y - y_m) \sin \theta_m] \\ Y &= \frac{1}{r_m} [-(x - x_m) \sin \theta_m + (y - y_m) \cos \theta_m] \end{aligned} \right\} \quad (2.1)$$

The local equation of the ellipse ∂D_m , i.e. the boundary of D_m , has the form

$$\frac{X^2}{(1 + \alpha)^2} + \frac{Y^2}{(1 - \alpha)^2} = 1. \quad (2.2)$$

The foci of the ellipse ∂D_m in the local coordinates are located at $(\pm 2\sqrt{\alpha}, 0)$.

Introduce the local and global complex coordinates $Z = X + iY$ and $z = x + iy$, where i denotes the imaginary unit. The Joukowski conformal mapping

$$Z = w + \frac{\alpha}{w} \quad (2.3)$$

transforms the annulus $\sqrt{\alpha} < |w| < 1$ onto $D_m - \Gamma_m$, where Γ_m is the slit $(-2\sqrt{\alpha}, 2\sqrt{\alpha})$ along the X -axis. The inverse mapping to equation (2.3) has the form

$$w = \frac{1}{2} \left(Z + \sqrt{Z^2 - 4\alpha} \right), \quad (2.4)$$

where the branch of the square root is chosen in such a way that

$$\lim_{Z \rightarrow X \pm i0} \sqrt{Z^2 - 4\alpha} = \pm i\sqrt{4\alpha - X^2} \quad \text{for } -2\sqrt{\alpha} < X < 2\sqrt{\alpha}. \quad (2.5)$$

Formulae (2.3) and (2.4) in the global coordinates become

$$z = s_m \left(w + \frac{\alpha}{w} \right) + a_m \quad (2.6)$$

and

$$w = \frac{1}{2} \left[\frac{z - a_m}{s_m} + \sqrt{\left(\frac{z - a_m}{s_m} \right)^2 - 4\alpha} \right], \quad (2.7)$$

where $a_m = x_m + iy_m$ and $s_m = r_m e^{i\theta_m}$.

(b) Conjugation conditions

Let D denote the complement of the closures of all domains D_m to the plane. It is convenient to assume that infinity belongs to D as it is usually done in complex analysis. We study the conductivity of the two-dimensional composite, when the domains D and D_m are occupied by materials of unit and λ conductivity, respectively, where $0 < \lambda < \infty$. Then, the potentials $u(z)$ and

$u_m(z)$ are harmonic in D and D_m ($m = 1, 2, \dots, n$) and satisfy the conjugation (transmission) conditions

$$u = u_m, \quad \frac{\partial u}{\partial n} = \lambda \frac{\partial u_m}{\partial n} \text{ on } \partial D_m, \quad m = 1, 2, \dots, n, \quad (2.8)$$

where $\partial/\partial n$ is the outward normal derivative to the ellipses. For simplicity, it is assumed that the potential $u(z)$ has singularities only in the domain D described by a function $\operatorname{Re} f(z)$, where $f(z)$ is analytic in all the inclusions D_k and Re stands for the real part of a complex number. For instance, if $u(z) \sim x = \operatorname{Re} z$ at infinity, i.e.

$$f(z) = z, \quad (2.9)$$

the external flux is applied parallel to the x -axis.

Following Mityushev & Rogosin (2000), introduce complex potentials $\varphi(z)$ and $\varphi_m(z)$ analytic in D and D_m , respectively, in such a way that $u(z)$ and $u_m(z)$ are related to the complex potentials, respectively, by

$$u(z) = \operatorname{Re} [\varphi(z) + z] \quad \text{and} \quad u_m(z) = \frac{2}{1 + \lambda} \operatorname{Re} \varphi_m(z). \quad (2.10)$$

Then the conditions (2.8) can be reduced to the following \mathbb{R} -linear problem

$$\varphi(t) = \varphi_k(t) - \overline{\rho \varphi_k(t)} - f(t), \quad t \in \partial D_k, \quad k = 1, 2, \dots, n, \quad (2.11)$$

where the overbar stands for the complex conjugation.

(c) Integral equations

In this section, we reduce the problem (2.11) to integral equations. First, recall the Sokhotskij–Plemelj formulae. The curve $L := \cup_{k=1}^n \partial D_k$ divides the complex plane onto two domains D^+ and D . Here, each curve ∂D_k is orientated in the clockwise direction. Let $\mu(t)$ be a Hölder continuous function on L . Then the function

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\mu(t)}{t - z} dt \quad (2.12)$$

is analytic and continuous on the complex plane except L where its limit boundary values $\Phi^+(t) = \lim_{z \rightarrow t, z \in D^+} \Phi(z)$ and $\Phi^-(t) = \lim_{z \rightarrow t, z \in D} \Phi(z)$ satisfy the jump condition

$$\Phi^+(t) - \Phi^-(t) = \mu(t), \quad t \in L. \quad (2.13)$$

The relation (2.11) can be written in the form (2.13) with $\Phi^+(t) = \varphi_k(t) - f(t)$, $\Phi^-(t) = \varphi(t)$ and $\mu(t) = \rho \overline{\varphi_k(t)}$. Then equation (2.11) yields

$$\varphi_k(z) = \rho \sum_{m=1}^n \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m(t)}}{t - z} dt + f(z), \quad z \in D_k, \quad k = 1, 2, \dots, n \quad (2.14)$$

and

$$\varphi(z) = \rho \sum_{m=1}^n \frac{1}{2\pi i} \int_{\partial D_m} \frac{\overline{\varphi_m(t)}}{t - z} dt + f(z), \quad z \in D. \quad (2.15)$$

One can consider equation (2.14) as a system of integral equations with respect to $\varphi_k(z)$ analytic in D_k and continuously differentiable in its closure. When $\varphi_k(z)$ are found, $\varphi(z)$ is calculated by equation (2.15). It is worth noting that equations (2.14) are not the classic integral equations of the potential theory. They correspond to integral equations that can be deduced from the Schwarz alternating method due to Mityushev & Rogosin (2000).

Following Mityushev & Rogosin (2000), one can prove that the system of integral equations (2.14) has a unique solution up to an additive constant when $|\rho| < 1$ that corresponds to the physical meaning of the problem (see equation (1.1)). This solution can be found by the method of successive approximations uniformly convergent in any compact subset of D^+ . This theoretical result will be proved in detail in a separate paper. In this paper, attention is paid to finding a constructive solution to equations (2.14).

3. Solution to integral equations

Integral equations (2.14) are deduced for general shape of the inclusions D_k and can be numerically solved, for instance, by the boundary element method due to Lifanov (1996). This method and other numerical methods are effective for numerically given geometries and data. However, if one wishes to preserve the contrast parameter ρ , the concentration ν and parameters of the shape D_k in symbolic form, then he should apply symbolic computations.

In this section, equations (2.14) are solved for the elliptical shape of inclusions with the parameters ρ , α , θ_k , a_k and n in symbolic form. First, the integral equations (2.14) on the complex plane z are transformed to equations on the complex plane w . The latter equations are reduced to functional equations. A simple constructive algorithm to solve the functional equations in symbolic form is described.

(a) Transformation of integral equations

Let k be fixed in equations (2.14). The doubly connected domain $D_k - \Gamma_k$ is mapped onto the annulus $\sqrt{\alpha} < |w| < 1$ by the conformal mapping (2.7); D_k is transformed onto the unit circle $|w|=1$, and Γ_k onto the circle $|w|=\sqrt{\alpha}$. Introduce the functions

$$\Phi_m(w) = \varphi_m(z) = \varphi_m \left[s_m \left(w + \frac{\alpha}{w} \right) + a_m \right] \quad (3.1)$$

analytic in $\sqrt{\alpha} < |w| < 1$ and continuous in $\sqrt{\alpha} \leq |w| \leq 1$. Substitute equation (3.1) in equations (2.14) and change the variables in the integrals as follows:

$$t = s_m \left(\tau + \frac{\alpha}{\tau} \right) + a_m. \quad (3.2)$$

Then equations (2.14) become

$$\Phi_k(w) = \rho \sum_{m=1}^n \frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\Phi_m(\tau)} (1 - \alpha/\tau^2) d\tau}{\tau + \alpha/\tau - (s_k/s_m)(w + \alpha/w) + (a_m - a_k)/s_m} + F(w),$$

$$\sqrt{\alpha} < |w| < 1, \quad k = 1, 2, \dots, n, \quad (3.3)$$

where $F(w) = f(z)$.

Moreover, it follows from the continuity of $\varphi_k(z)$ when z passes the slit Γ_k that

$$\Phi_k(\tau) = \Phi_k\left(\frac{\alpha}{\tau}\right), \quad |\tau| = \sqrt{\alpha}. \quad (3.4)$$

Equation (3.4) implies that $\Phi_k(w)$ is represented in the form

$$\Phi_k(w) = \phi_k(w) + \phi_k\left(\frac{\alpha}{w}\right), \quad \sqrt{\alpha} \leq |w| \leq 1, \quad (3.5)$$

where $\phi_k(w)$ is analytic in the unit disc $|w| < 1$. Equation (3.5) follows from the representation of $\Phi_k(w)$ in the form of the Laurent series in the annulus $\sqrt{\alpha} \leq |w| \leq 1$ and from the application of equation (3.4). The same arguments yield the representation of $F(w)$ in the form $F(w) = g_k(w) + g_k(\alpha/w)$, where $g_k(w)$ is analytic in the unit disc. Substitution of equation (3.5) into equation (3.3) yields

$$\begin{aligned} \phi_k(w) + \phi_k\left(\frac{\alpha}{w}\right) &= \rho \sum_{m=1}^n [P_{km}(w) + Q_{km}(w)] + g_k(w) + g_k\left(\frac{\alpha}{w}\right), \\ \sqrt{\alpha} < |w| < 1, \quad k &= 1, 2, \dots, n, \end{aligned} \quad (3.6)$$

where

$$P_{km}(w) = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\phi_m(1/\bar{\tau})(1-\alpha/\tau^2)} d\tau}{\tau + \alpha/\tau - (s_k/s_m)(w + \alpha/w) + (a_m - a_k)/s_m} \quad (3.7)$$

and

$$Q_{km}(w) = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\phi_m(\alpha\bar{\tau})(1-\alpha/\tau^2)} d\tau}{\tau + \alpha/\tau - (s_k/s_m)(w + \alpha/w) + (a_m - a_k)/s_m}. \quad (3.8)$$

Here, the relation $\tau = 1/\bar{\tau}$ on the unit circle is used.

(b) Calculation of the integrals

In this section, the integrals (3.7) and (3.8) are analytically calculated by residues. To study singularities of the integrands, we investigate the roots of the denominator. We have the quadratic equation with respect to τ ,

$$\tau^2 - s_m^{-1} \left[s_k \left(w + \frac{\alpha}{w} \right) + a_k - a_m \right] \tau + \alpha = 0. \quad (3.9)$$

The cases of equal and non-equal k and m have to be separately investigated.

(i) Let $k = m$. Then equation (3.9) becomes

$$\tau^2 - \left(w + \frac{\alpha}{w} \right) \tau + \alpha = 0. \quad (3.10)$$

Its two solutions have the form

$$\tau_1 = w \quad \text{and} \quad \tau_2 = \frac{\alpha}{w}. \quad (3.11)$$

One can see τ_1 and τ_2 lie in the unit disc, as $\sqrt{\alpha} < |w| < 1$.

(ii) Let $k \neq m$. To avoid confusion with equation (3.11), the roots of equation (3.9) in this case are denoted by w_1 and w_2 :

$$\left. \begin{aligned}
 w_1 &= \frac{1}{2} \left\{ s_m^{-1} \left[s_k \left(w + \frac{\alpha}{w} \right) + a_k - a_m \right] \right. \\
 &\quad \left. - \sqrt{\left[\frac{s_k \left(w + \alpha/w \right) + a_k - a_m}{s_m} \right]^2 - 4\alpha} \right\} \\
 \text{and} \\
 w_2 &= \frac{1}{2} \left\{ s_m^{-1} \left[s_k \left(w + \frac{\alpha}{w} \right) + a_k - a_m \right] \right. \\
 &\quad \left. + \sqrt{\left[\frac{s_k \left(w + \alpha/w \right) + a_k - a_m}{s_m} \right]^2 - 4\alpha} \right\}
 \end{aligned} \right\}. \tag{3.12}$$

The branch of the square root is chosen in accordance with equation (2.5). We will prove that $|w_1| < 1$ and $|w_2| > 1$. Consider equation (3.9) in the domain $|\tau| \geq 1$. Let D_m^- denote the complement of D_m to the extended complex plane z . Changing the variable τ by t (see equation (3.2)), we obtain

$$t = z \tag{3.13}$$

with respect to $t \in D_m^- \cup \partial D_m$, where the parameter

$$z = s_k \left(w + \frac{\alpha}{w} \right) + a_k \tag{3.14}$$

belongs to D_k . Therefore, equation (3.13) has the unique solution $t = z \in D_m^-$, as $D_k \subset D_m^-$ ($k \neq m$). Hence, equation (3.9) has a unique solution in the domain $|\tau| \geq 1$. The second root w_2 from equation (3.12) in the variables τ and w corresponds to the root $t = z$ in the variables t and z . Therefore, $|w_2| \geq 1$. It follows from Viète’s formulae that $w_1 w_2 = \alpha$. This implies that $|w_1| < 1$, as $\alpha < 1$.

We now proceed to analytically calculate the integral (3.7). For $m \neq k$, it can be written in the form

$$P_{km}(w) = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{\overline{\phi_m(1/\bar{\tau})}(\tau^2 - \alpha) d\tau}{\tau(\tau - w_1)(\tau - w_2)}, \tag{3.15}$$

where w_1 and w_2 have the form (3.12). Calculate $P_{km}(w)$ by residua using the analyticity of $\overline{\phi_m(1/\bar{\tau})}$ in $|\tau| > 1$,

$$\begin{aligned}
 P_{km}(w) &= -\text{res}_\infty \frac{\overline{\phi_m(1/\bar{\tau})}(\tau^2 - \alpha)}{\tau(\tau - w_1)(\tau - w_2)} - \text{res}_{w_2} \frac{\overline{\phi_m(1/\bar{\tau})}(\tau^2 - \alpha)}{\tau(\tau - w_1)(\tau - w_2)} \\
 &= \overline{\phi_m(0)} - \overline{\phi_m \left(\frac{w_1}{\alpha} \right)} \quad (m \neq k),
 \end{aligned} \tag{3.16}$$

where the relation $w_1 w_2 = \alpha$ is used. Similar arguments yield formulae

$$\left. \begin{aligned} P_{kk}(w) &= \overline{\phi_k(0)}, \\ Q_{km}(w) &= \overline{\phi_m(0)} - \overline{\phi_m(\alpha w_1)} \quad (m \neq k) \\ \text{and} \quad Q_{kk}(w) &= -\overline{\phi_k(0)} + \overline{\phi_k(\alpha \bar{w})} + \overline{\phi_k\left(\frac{\alpha^2}{w}\right)}. \end{aligned} \right\} \quad (3.17)$$

Here, the following formula for $\Phi^-(w)$ analytic in $|w| > 1$ is used due to Gakhov (1966):

$$\frac{1}{2\pi i} \int_{|\tau|=1} \frac{\Phi^-(\tau)}{\tau - w} d\tau = -\Phi^-(w) + \Phi^-(\infty).$$

(c) *Functional equations*

The integrals (3.7) and (3.8) have been exactly calculated in the previous section. Substituting the results (3.16) and (3.17) into equation (3.6), we transform the integral equations (3.6) to the following functional equations:

$$\begin{aligned} \phi_k(w) + \phi_k\left(\frac{\alpha}{w}\right) &= \rho \left\{ \overline{\phi_k(\alpha \bar{w})} + \overline{\phi_k\left(\frac{\alpha^2}{w}\right)} - \sum_{m \neq k} \left[-2\overline{\phi_m(0)} + \overline{\phi_m(\alpha \beta_{km}(w))} \right. \right. \\ &\quad \left. \left. + \overline{\phi_m\left(\frac{\beta_{km}(w)}{\alpha}\right)} \right] + g_k(w) + g_k\left(\frac{\alpha}{w}\right) \right\}, \\ \sqrt{\alpha} < |w| < 1, \quad k &= 1, 2, \dots, n. \end{aligned} \quad (3.18)$$

Here, for convenience the root w_1 is written as the function of w ,

$$\beta_{km}(w) = \frac{1}{2} \left\{ s_m^{-1} \left[s_k \left(w + \frac{\alpha}{w} \right) + a_k - a_m \right] - \sqrt{h_{km}(w)} \right\}, \quad (3.19)$$

where

$$h_{km}(w) = \left[\frac{s_k(w + \alpha/w) + a_k - a_m}{s_m} \right]^2 - 4\alpha. \quad (3.20)$$

The left-hand part of equation (3.18) consists of the functions $\phi_k(w)$ and $\phi_k(\alpha/w)$ analytic in $|w| < 1$ and $|w| > \sqrt{\alpha}$, respectively. Denote by P^+ the project operator that transforms a function analytic in $\sqrt{\alpha} < |w| < 1$ to its part analytic in the unit disc. This operator can be considered as taking the regular part of the Laurent series or as the integral operator $(1/2\pi i) \int_{|\zeta|=1} \bullet dw/(w - \zeta)$ with $|\zeta| < 1$.

Application of P^+ to equation (3.18) yields

$$\phi_k(w) + \phi_k(0) = \rho \left\{ \overline{\phi_k(\alpha \bar{w})} + \overline{\phi_k(0)} - \sum_{m \neq k} \left[-2\overline{\phi_m(0)} + P^+ \overline{\phi_m(\alpha \beta_{km}(w))} + P^+ \overline{\phi_m\left(\frac{\beta_{km}(w)}{\alpha}\right)} \right] \right\} + g_k(w) + g_k(0), \quad |w| \leq 1, \quad k=1, 2, \dots, n. \quad (3.21)$$

Here, the relation $P^+ \overline{\phi_k(\alpha^2/\bar{w})} = \overline{\phi_k(0)}$ is used.

If the external field is applied parallel to the x -axis, then equation (2.9) implies that $F(w) = z = s_k(w + \alpha/w) + a_k$, hence

$$g(w) = s_k w + a_k. \quad (3.22)$$

Hereafter, equations (3.21) are considered with the function (3.22).

One can consider equations (3.21) as a system of functional equations with respect to $\phi_k(w)$ analytic in the unit disc and continuous in its closure. Despite equations (3.21) involving the operator P^+ , they are more convenient in symbolic computations than equations (3.18). As stated at the end of §2*c*, the system (3.21) has a unique solution. This solution can be found by the method of successive approximations. Equations (3.21) can be considered as iterative functional equations with shift into domain (Kuczma *et al.* 1990), as $|\beta_{km}(w)| < 1$. The later inequality is equivalent to the relation $|w_1| < 1$ justified in §3*b*.

The special case of $\alpha = 0$ can be investigated by the limit $\alpha \rightarrow 0$ when the ellipses become circles. In this case we arrive at the functional equations discussed in Mityushev (1993). Applications of such equations and their modifications to determining the effective conductivity tensor of a composite containing circular inclusions were presented by Berlyand & Mityushev (2001, 2005) and Mityushev *et al.* (2008). Discussion of dilute regime and regular locations of circular inclusions was presented by Milton (2002), McPhedran (1986) and McPhedran & Milton (1987) by application of Rayleigh's method.

4. Effective conductivity tensor Λ

(a) General formulae for Λ

The effective conductivity tensor

$$\Lambda = \begin{pmatrix} \lambda^x & \lambda^{xy} \\ \lambda^{xy} & \lambda^y \end{pmatrix} \quad (4.1)$$

can be calculated by the Maxwell approach. Following Mityushev (1999, 2005), one can obtain formula for dilute composites

$$\lambda^x - i\lambda^{xy} = 1 + 2\rho v \frac{1}{n} \sum_{k=1}^n I_k + O(v^2), \quad (4.2)$$

where I_k denotes the double integral

$$I_k = \frac{1}{|D_k|} \int_{D_k} \varphi'_k(z) \, dx \, dy, \quad (4.3)$$

where $\varphi_k(z)$ is a solution of equation (2.11) with $f(z) = z$. The component λ^y can be calculated by the formula due to Mityushev (2001):

$$\lambda^y + i\lambda^{xy} = 1 + 2\rho v \frac{1}{n} \sum_{k=1}^n I_k + O(v^2), \quad (4.4)$$

where the double integral I_k is calculated with $\varphi_k(z)$, a solution of equation (2.11) with $f(z) = -iz$. The later corresponds to the external field applied in the y -direction.

The integral I_k can be transformed by Green's formula written in complex form

$$I_k = -\frac{1}{2i|D_k|} \int_{\partial D_k} \varphi_k(t) \, d\bar{t}, \quad (4.5)$$

where $|D_k| = r_k^2(1 - \alpha^2)$ is the area of D_k . The integral (4.5) can be calculated by the change of variables (see equation (3.2)). Then

$$\bar{t} = \bar{s}_k \left(\bar{\tau} + \frac{\alpha}{\bar{\tau}} \right) + \bar{a}_k.$$

Using the relation $\bar{\tau} = 1/\tau$ on the unit circle, we obtain

$$d\bar{t} = \bar{s}_k \left(\alpha - \frac{1}{\tau^2} \right) d\tau.$$

Substituting equations (3.1) and (3.5) into equation (4.5) and calculating the integrals by the residues (at zero and at infinity), we arrive at the simple formula

$$I_k = s_k^{-1} \overline{\phi'_k(0)}. \quad (4.6)$$

Therefore, equation (4.2) becomes

$$\lambda^x - i\lambda^{xy} = 1 + 2\rho v \frac{1}{n} \sum_{k=1}^n s_k^{-1} \phi'_k(0) + O(v^2). \quad (4.7)$$

The value $\phi'_k(0)$ can be computed after solution to functional equations (3.21). It is noted at the end of §3b that these equations can be solved by the method of successive approximations. Differentiating equations (3.21), multiplying the result by s_k^{-1} and introducing the function $\psi_k(w) = s_k^{-1} \phi'_k(w)$, then it implies that

$$\begin{aligned} \psi_k(w) = & \rho \alpha e^{-2i\theta_k} \overline{\psi_k(\alpha \bar{w})} - \rho \sum_{m \neq k} \frac{\bar{s}_m}{s_k} P^+ \left[\overline{\alpha \psi_m(\alpha \beta_{km}(w))} \right. \\ & \left. + \frac{1}{\alpha} \overline{\psi_m \left(\frac{\beta_{km}(w)}{\alpha} \right)} \right] \beta'_{km}(w) + 1, \quad |w| \leq 1, \quad k = 1, 2, \dots, n. \end{aligned} \quad (4.8)$$

Here, the commutativity of the operators P^+ and d/dw is used. Formula (4.7) becomes

$$\lambda^x - i\lambda^{xy} = 1 + 2\rho v \frac{1}{n} \sum_{k=1}^n \psi_k(0) + O(v^2). \quad (4.9)$$

Therefore, to determine λ^x and λ^{xy} , one can solve the functional equation (4.8), compute $\psi_k(0)$ and substitute it in equation (4.9).

(b) *Algorithm to compute Λ in symbolic form*

Introducing the dimensionless parameter $\ell = \max_{k \neq m} \sqrt{r_k r_m} |a_k - a_m|^{-1}$ characterizes the ratio of the line sizes of inclusions to the distances between them. To describe the structure of the functional equation (4.8), we shortly write it in the form

$$\psi_k(w) = \rho\alpha e^{-2i\theta_k} \overline{\psi_k(\alpha\bar{w})} - \rho\ell^2 \sum_{m \neq k} (A_{km} \psi_m)(w) + 1, \quad |w| \leq 1, \quad k = 1, 2, \dots, n, \quad (4.10)$$

where A_{km} is a bounded linear operator. It will be clear later why the power ℓ^2 is taken as a coefficient. Equation (4.10) can be solved by the following two iterative schemes. First, the direct iterations can be applied to it to obtain

$$\begin{aligned} \psi_k^{(0)}(w) &= 1, & \psi_k^{(p+1)}(w) &= \rho\alpha e^{-2i\theta_k} \overline{\psi_k^{(p)}(\alpha\bar{w})} - \rho\ell^2 \sum_{m \neq k} (A_{km} \psi_m^{(p)})(w) + 1, \\ & & p &= 0, 1, 2, \dots, \end{aligned} \quad (4.11)$$

where $\psi_k^{(p)}(w)$ is the p th approximation of $\psi_k(w)$. As it is noted at the end of §3b, the iterations (4.11) always converge. The rate of the convergence is equal to $|\rho|$. Even in the case $|\rho| = 1$, the method (4.11) can be modified to be convergent as it was done by Mityushev & Rogosin (2000).

The second iterative scheme is constructed on the basis of the parameter ℓ^2 :

$$\begin{aligned} \psi_k^{(0)}(w) &= \rho\alpha e^{-2i\theta_k} \overline{\psi_k^{(0)}(\alpha\bar{w})} + 1, \\ \psi_k^{(p+1)}(w) &= \rho\alpha e^{-2i\theta_k} \overline{\psi_k^{(p+1)}(\alpha\bar{w})} - \rho\ell^2 \sum_{m \neq k} (A_{km} \psi_m^{(p)})(w) + 1, \quad p = 0, 1, 2, \dots \end{aligned} \quad (4.12)$$

Therefore, at each step the functional equation has to be solved

$$\psi_k^{(p)}(w) = \rho\alpha e^{-2i\theta_k} \overline{\psi_k^{(p)}(\alpha\bar{w})} + h_k^{(p)}(w), \quad |w| \leq 1. \quad (4.13)$$

Here, the function $h_k^{(p)}(w)$ is expressed through $\psi_k^{(p-1)}(w)$ computed at the previous step

$$h_k^{(p)}(w) = -\rho\ell^2 \sum_{m \neq k} (A_{km} \psi_m^{(p-1)})(w) + 1.$$

Mityushev & Rogosin (2000) proved that equation (4.13) has a unique solution represented in the form of the absolutely convergent series

$$\psi_k^{(p)}(w) = \sum_{q=0}^{\infty} (\rho\alpha)^{2q} \left[h_k^{(p)}(\alpha^{2q}w) + e^{-2i\theta_k} \rho\alpha h_k^{(p)}(\alpha^{2q+1}w) \right]. \quad (4.14)$$

The iterative scheme (4.12) corresponds to the generalized alternating Schwarz method discussed by Mityushev & Rogosin (2000).

The zero approximation of equation (4.12) can be easily obtained from equation (4.14):

$$\psi_k^{(0)}(w) = \frac{1 + \rho\alpha e^{-2i\theta_k}}{1 - \rho^2\alpha^2}. \quad (4.15)$$

Substitution of equation (4.15) into equation (4.9) yields the Maxwell approximation

$$\lambda^x - i\lambda^{xy} = 1 + \frac{2\rho v}{1 - \rho^2\alpha^2} \frac{1}{n} \sum_{k=1}^n (1 + \rho\alpha e^{-2i\theta_k}) + O(v^2). \quad (4.16)$$

Averaging over the orientation of the inclusions yields formula (1.4).

Higher approximations can be obtained by numerical and symbolic computations. It is worth noting that the realization of the iterative schemes (4.11) and (4.12) involves computations of the compositions of functions and of the regular parts of the Laurent series. There is not any integral that requires numerical computations that rather could not be computed in symbolic form.

(c) *First-order approximation for general distribution of inclusions*

We now proceed to determine $\psi_k^{(1)}(0)$ via $\psi_k^{(0)}(w)$ given by equation (4.15) from the functional equation

$$\psi_k^{(1)}(w) = \rho\alpha e^{-2i\theta_k} \overline{\psi_k^{(1)}(\alpha\bar{w})} + h_k^{(1)}(w), \quad |w| \leq 1, \quad (4.17)$$

where

$$h_k^{(1)}(w) = 1 - \rho \sum_{m \neq k} \frac{\overline{s_m}}{s_k} \frac{1 + \rho\alpha e^{-2i\theta_m}}{1 - \rho^2\alpha^2} \left(\alpha + \frac{1}{\alpha} \right) (P^+ \beta'_{km})(w). \quad (4.18)$$

The value $\psi_k^{(1)}(0)$ can be immediately determined by substitution of $w = 0$ into equation (4.17):

$$\psi_k^{(1)}(0) = \frac{h_k^{(1)}(0) + \rho\alpha e^{-2i\theta_k} \overline{h_k^{(1)}(0)}}{1 - \rho^2\alpha^2}. \quad (4.19)$$

Therefore, to determine $\psi_k^{(1)}(0)$, it is sufficient to calculate $(P^+\beta'_{km})(0)$. It is convenient to calculate $(\alpha + 1/\alpha)(P^+\beta'_{km})(0)$ in the form of the expansion on α ,

$$\left(\alpha + \frac{1}{\alpha}\right)(P^+\beta'_{km})(0) = -s_k s_m \left[\frac{1 + \alpha^2}{(a_k - a_m)^2} + 3\alpha \frac{s_k^2 + s_m^2}{(a_k - a_m)^4} + 10\alpha^2 \frac{s_k^4 + 3s_k^2 s_m^2 + s_m^4}{(a_k - a_m)^6} + \dots \right]. \quad (4.20)$$

This formula explicitly shows that the value $\psi_k^{(1)}(0)$ is of order ℓ^2 . Further, only the terms to $O(\ell^2)$ are kept in equation (4.20) and in the following computations. Substitution of equation (4.20) into equation (4.18) and of equation (4.19) into equation (4.9) yield

$$\lambda^x - i\lambda^{xy} = 1 + \frac{2\rho v}{1 - \rho^2 \alpha^2} \frac{1}{n} \sum_{k=1}^n [h_k^{(1)}(0) + \rho\alpha e^{-2i\theta_k} \overline{h_k^{(1)}(0)}] + O(v^2), \quad (4.21)$$

where

$$h_k^{(1)}(0) = 1 + \frac{\rho(1 + \alpha^2)}{1 - \rho^2 \alpha^2} \sum_{m \neq k} (1 + \rho\alpha e^{-2i\theta_m}) \frac{r_m^2}{(a_k - a_m)^2}. \quad (4.22)$$

Formula (4.21) is valid up to $O(v^2)$ and takes into account first-order interactions between the inclusions.

(d) Parallel inclusions

Consider equal elliptical inclusions with the parallel semi-axes. Then $\theta_k = \theta$ and $r_k = r$ for all $k = 1, 2, \dots, n$ and equations (4.22) and (4.21) are simplified, respectively, to

$$h_k^{(1)}(0) = 1 + \frac{\rho(1 + \alpha^2)(1 + \rho\alpha e^{-2i\theta})}{1 - \rho^2 \alpha^2} \sum_{m \neq k} \frac{\ell^2}{(a_k - a_m)^2} \quad (4.23)$$

and

$$\lambda^x - i\lambda^{xy} = 1 + \frac{2\rho v}{1 - \rho^2 \alpha^2} \frac{1}{n} \sum_{k=1}^n [h_k^{(1)}(0) + \rho\alpha e^{-2i\theta} \overline{h_k^{(1)}(0)}] + O(v^2). \quad (4.24)$$

Consider the case $\theta = 0$. Introduce the sum

$$\mathcal{S}_2(n) = \frac{1}{n} \sum_{k=1}^n \sum_{m \neq k} \frac{1}{(a_k - a_m)^2}. \quad (4.25)$$

Then the components of the effective conductivity tensor take the form

$$\left. \begin{aligned} \lambda^x &= 1 + \frac{2\rho v}{1 - \rho\alpha} \left[1 + \frac{\rho(1 + \alpha^2)}{1 - \rho\alpha} \ell^2 \operatorname{Re} \mathcal{S}_2(n) \right] + O(v^2) \\ \text{and} \\ \lambda^{xy} &= -\frac{2\rho^2 v(1 + \alpha^2)}{(1 - \rho\alpha)^2} \ell^2 \operatorname{Im} \mathcal{S}_2(n) + O(v^2), \end{aligned} \right\} \quad (4.26)$$

where Re and Im stand for the real and the imaginary parts of complex values, respectively. To calculate λ^y , it is sufficient to replace a_k by ia_k . Then in accordance with equation (4.4), we obtain

$$\lambda^y = 1 + \frac{2\rho\nu}{1 - \rho\alpha} \left[1 - \frac{\rho(1 + \alpha^2)}{1 - \rho\alpha} \ell^2 \text{Re } \mathcal{S}_2(n) \right] + O(\nu^2). \quad (4.27)$$

(e) *Randomly distributed inclusions*

Consider equal elliptical inclusions randomly distributed on the plane. More precisely, it is assumed that all $r_k = r$ are fixed and each angle θ_k is a random value uniformly distributed on $[0, \pi]$. Moreover, the centres a_k are independent identically distributed random variables such that the random vector (a_1, a_2, \dots, a_n) obeys a non-overlapping distribution for ellipses. The distribution of the centres and of the angles are mutually independent. Then the averages over θ_k from equation (4.21) vanish

$$\langle e^{-2i\theta_k} \rangle = 0 \quad \text{and} \quad \langle e^{2i(\theta_m - \theta_k)} \rangle = 0 \quad (4.28)$$

and equation (4.21) becomes

$$\langle \lambda^x - i\lambda^{xy} \rangle = 1 + \frac{2\rho\nu}{1 - \rho^2\alpha^2} \left[1 + \frac{\rho(1 + \alpha^2)}{1 - \rho^2\alpha^2} \ell^2 \langle \mathcal{S}_2(n) \rangle \right] + O(\nu^2). \quad (4.29)$$

Formula (4.29) is a generalization of the Maxwell formula (1.4). The term $\langle \mathcal{S}_2(n) \rangle$ describes the first-order interactions of the inclusions.

(f) *Infinite number of inclusions*

The most interesting begins when n tends to infinity. Then the first-order term in ℓ^2 can be transformed into the first-order approximation in the concentration ν . The following limit of equation (4.25) plays the main role in the investigation

$$\mathcal{S}_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{m \neq k}^n \frac{1}{(a_k - a_m)^2}. \quad (4.30)$$

The formal pass to an infinite sum transforms equation (4.30) to a conditionally convergent series. To treat this series, fix an order of $|a_k|$. Let $0 < |a_1| \leq |a_2| \leq \dots \leq |a_k| \leq \dots$. The limit (4.30) was written via the absolutely convergent sums by Mityushev (1999)

$$\begin{aligned} \mathcal{S}_2 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{m=1}^{\infty'} \left[\frac{1}{(a_k - a_m)^2} - \frac{1}{a_m^2} \right] \\ &+ \lim_{\Delta z \rightarrow \infty} \sum_{k=1}^{\infty} \left[\frac{1}{a_k^2} - \frac{1}{(z^* - a_k)(z^* - a_k + \Delta z)} \right], \end{aligned} \quad (4.31)$$

where $m \neq k$ in the sum $\sum_{m=1}^{\infty'}$. The second limit from equation (4.31) is well defined, if it does not depend on the choice of point z^* and on the curve along which $\Delta z \rightarrow \infty$.

Mityushev (1999) proved that the first limit $\lim_{n \rightarrow \infty}$ from equation (4.31) exists, if the area fraction

$$\nu = \lim_{R \rightarrow \infty} \frac{A(R)}{\pi R^2} \quad (4.32)$$

exists. Here $A(R)$ is the area of inclusions located in a disc of radius R . The limit (4.32) has to be independent on the centre of the large disc. Hence, the first term from equation (4.32) is defined as the average of the absolutely convergent sums $\sum_{m=1}^{\infty} a_m'$. It is forbidden to interchange the limit $\lim_{\Delta z \rightarrow \infty}$ and the sum $\sum_{k=1}^{\infty} a_k^{-2}$, because, in general, this produces the conditionally convergent series $\sum_{k=1}^{\infty} a_k^{-2}$. Therefore, formula (4.31) cannot be simplified in general cases. The limits from equation (4.31) exist under the above described conditions on the distribution of a_k . This is not surprising because the composite discussed may not be homogenized.

The above theoretical discussion concerning \mathcal{S}_2 is simplified, if we consider a doubly periodic distribution of a_k . Then the considered composite is homogenized and the limit (4.31) always exists. Let N points a_k ($k = 1, 2, \dots, N$) are placed in a rectangular periodicity cell Q with the sides γ and γ^{-1} . The area of the cell holds 1. All other points a_k are obtained from the first N points by translations on the vectors γ and $i\gamma^{-1}$ expressed in the form of complex numbers. Further, it is convenient to apply to the limit (4.30) the Eisenstein summation (for detail, see Mityushev *et al.* 2008). Then equation (4.30) becomes

$$\mathcal{S}_2 = \frac{1}{N} \sum_{k=1}^N \sum_{m \neq k} \frac{1}{(a_k - a_m)^2} = \frac{1}{N} \sum_{k,m} E_2^*(a_k - a_m), \quad (4.33)$$

where $E_2(z)$ denotes the Eisenstein function, $E_2^*(z)$ coincides with $E_2(z)$ for $z \neq 0$ and $E_2^*(0) := 2\gamma^{-1}\zeta(\gamma/2)$. Here, ζ is the Weierstrass ζ -function corresponding to the periodicity cell Q . In the case $N = 1$, we have $\mathcal{S}_2 = S_2 = 2\gamma^{-1}\zeta(\gamma/2)$, where S_2 stands for the standard lattice sum when all points a_k are produced from $z = 0$ by translations on the fundamental lattice vectors. In the case of the square cell, $\mathcal{S}_2 = S_2 = \pi$.

Substitution of equation (4.33) instead of $\mathcal{S}_2(n)$ into equation (4.26) and simple transformations up to $O(v^3)$ yield

$$\left. \begin{aligned} \lambda^x &= 1 + \frac{2\rho\nu}{1 - \rho(\alpha + \nu((1 + \alpha^2)/(1 - \alpha^2)))N^{-1}\text{Re } \mathcal{S}_2/\pi} + O(v^3), \\ \text{and} \\ \lambda^{xy} &= -\frac{2\rho^2\nu^2}{(1 - \rho\alpha)^2} \frac{1 + \alpha^2}{1 - \alpha^2} \frac{N^{-1}\text{Im } \mathcal{S}_2}{\pi} + O(v^3). \end{aligned} \right\} \quad (4.34)$$

Using relations between the sums calculated in the perpendicular direction, we obtain

$$\lambda^y = 1 + \frac{2\rho\nu}{1 - \rho(\alpha + \nu((1 + \alpha^2)/(1 - \alpha^2)))(2\pi - N^{-1}\text{Re } \mathcal{S}_2)/\pi} + O(v^3). \quad (4.35)$$

In the case $N = 1$, we have $\mathcal{S}_2 = S_2$. The value $S_2 = 2\gamma^{-1}\zeta(\gamma/2)$ is a real number for a rectangular array (see Rylko (2000)). For small α , formulae (4.34) and (4.35) are coincided to equations (1.9).

Along similar lines, equation (4.29) implies that

$$\langle \lambda^x - i\lambda^{xy} \rangle = 1 + \frac{2\rho v}{1 - \rho^2\alpha^2 - \rho v((1 + \alpha^2)/(1 - \alpha^2))N^{-1}S_2/\pi} + O(v^3) \quad (4.36)$$

for randomly distributed ellipses with major semi-axes parallel to the x -axis. For the square array, $N = 1$ and $S_2 = \pi$. Then equation (4.36) becomes

$$\lambda_e = 1 + \frac{2\rho v}{1 - \rho^2\alpha^2 - \rho v(1 + \alpha^2)/(1 - \alpha^2)} + O(v^3). \quad (4.37)$$

A discrepancy between equations (1.9) and (4.34) with $N = 1$ can be explained in terms of the iterative schemes (4.11) and (4.12). Rayleigh's method is based on representations of the unknown functions and on the addition theorems that produce an infinite system of linear algebraic equations. Analysis of Rayleigh's infinite system for circular inclusions made by Rylko (2000) shows that Rayleigh's method corresponds to the iterative scheme (4.11) with $\alpha = 0$, i.e., Rayleigh's method presented in continuous form via functional equations yields a series on ρ . In this section, we use the iterative scheme (4.12) that produces, in general, other approximate formulae.

Remark 4.1. The above formulae are derived for any λ satisfying the inequality $0 < \lambda < \infty$. One can pass to the limit cases $\lambda = 0$ and $\lambda = \infty$ by direct substitution of $\rho = \pm 1$ in equations (4.34)–(4.37).

5. Conclusion and discussion

Exact and approximate formulae for the effective conductivity tensor Λ of a two-dimensional medium containing elliptical inclusions are important in many applications. In this paper, we specialize the terminology of electric or thermal conductivity to make the discussion more concrete.

The Maxwell approach concerning the dilute regime yields various approximate formulae for Λ presented in §1. These formulae are valid up to $O(v^2)$, where each inclusion independently influences the macroscopic properties of the medium. According to the homogenization theory, higher approximations require the study of a medium with infinitely many inclusions, in particular, periodically arranged. Such higher approximations would involve interactions between inclusions written via the lattice sums, the correlation functions, etc.

In this paper, the method of the functional equation is developed to boundary value problems with elliptical inclusions. The investigation is confined to the case when the first-order interactions between inclusions are taken into account. New analytical approximate formulae for Λ are derived. These formulae take into account the shape, the sizes and the location of the inclusions. The influence of the geometry is expressed by the sum (4.25), by the limit (4.30) and by the sum (4.33) for doubly periodic composites. The final formulae (4.26), (4.27) and (4.34)–(4.37) for Λ are valid up to $O(v^3)$. Formulae (4.21) and (4.22) are universal for any distribution of the centres of inclusions. One can directly consider them from the statistical point of view as is done in §4e. Formulae (4.34)–(4.37) can also be treated as a statistical result. Theoretically, this procedure is equivalent to

estimate the pair correlation function $P(z|0)$ discussed by Milton (2002). From a practical point of view, it is convenient instead of $P(z|0)$ (or of \mathcal{S}_2 given by equation (4.30)) to estimate \mathcal{S}_2 by equation (4.33) for sufficiently large N . It is also possible to investigate the expression in the right-hand side of equation (4.33) and formulae involving this expression for any N in symbolic form. After, one can put N infinitely large as it was done by Berlyand & Mityushev (2001, 2005) for circular inclusions.

Formulae (4.34)–(4.37) explicitly express the dependence of the effective conductivity tensor on the centres a_k of inclusions. If a_k are located at the sites of a regular array, \mathcal{S}_2 from equation (4.33) becomes the standard lattice sum S_2 and we arrive at formulae by McPhedran & Nicorovici (1997), Nicorovici & McPhedran (1996) and Yardley *et al.* (1999, 2001) deduced for special regular locations of ellipses. It is worth noting that the later papers contain higher-order approximations in ν for a regular array of ellipses. The method proposed in this paper can also be applied to higher-order approximations for an arbitrary distribution of inclusions. This can be done for numerically fixed a_k . However, direct computations with a_k in symbolic form yield enormous formulae that should be written in a simple form via the generalized lattice sums \mathcal{S}_k ($k \geq 2$). This requires advanced symbolic computations. Such computations were performed for circular inclusions ($\alpha = 0$) by Mityushev (2001), Mityushev *et al.* (2008) and Berlyand & Mityushev (2001, 2005) with a practically arbitrary precision in ν . The obtained analytical formulae are consistent with the results presented by Milton (2002), McPhedran (1986) and McPhedran & Milton (1987). One can believe that such computations could be developed to the general case of elliptical inclusions to derive higher-order formulae for the tensor Λ .

I am grateful to referees for remarks improving the presentation of the results.

References

- Ammari, H., Kang, H. & Touibi, K. 2005 Boundary layer techniques for deriving the effective properties of composite materials. *Asymptot. Anal.* **41**, 119–140.
- Bergman, D. J. 1978 The dielectric constants of a composite material—a problem in classical physics. *Phys. Rep. C* **43**, 377–407.
- Berlyand, L. & Mityushev, V. 2001 Generalized Clausius–Mossotti formula for random composites with circular fibers. *J. Stat. Phys.* **102**, 115–145. (doi:10.1023/A:1026512725967)
- Berlyand, L. & Mityushev, V. 2005 Increase and decrease of the effective conductivity of two-phase composites due to polydispersity. *J. Stat. Phys.* **118**, 481–509. (doi:10.1007/s10955-004-8818-0)
- Cohen, R. W., Cody, G. D., Coutts, M. D. & Abeles, B. 1973 Optical properties of granular silver and gold films. *Phys. Rev. B* **8**, 3689–3701. (doi:10.1103/PhysRevB.8.3689)
- Crowdy, D. G. 2008 The Schwarz problem in multiply connected domains and the Schottky–Klein prime function. *Complex Var. Elliptic Equations* **53**, 1–16. (doi:10.1080/17476930701682897)
- Gakhov, F. D. 1966 *Boundary value problems*. Oxford, UK: Pergamon Press/Addison-Wesley.
- Galeener, F. L. 1971a Submicroscopic-void resonance: the effect of internal roughness on optical absorption. *Phys. Rev. Lett.* **27**, 421–423. (doi:10.1103/PhysRevLett.27.421)
- Galeener, F. L. 1971b Submicroscopic-void resonance: the effect of internal roughness on optical absorption. *Phys. Rev. Lett.* **27**, 769. (doi:10.1103/PhysRevLett.27.769)
- Galeener, F. L. 1971c Optical evidence for a network of cracklike voids in amorphous germanium. *Phys. Rev. Lett.* **27**, 1716–1719. (doi:10.1103/PhysRevLett.27.1716)
- Garboczi, E. J. & Douglas, J. F. 1996 Intrinsic conductivity of objects having arbitrary shape and conductivity. *Phys. Rev. E* **53**, 6169–6180. (doi:10.1103/PhysRevE.53.6169)

- Hetherington, J. H. & Thorpe, M. F. 1992 The conductivity of a sheet containing inclusions with sharp corners. *Proc. R. Soc. Lond. A* **438**, 591–604. (doi:10.1098/rspa.1992.0128)
- Kuczma, M., Choczewski, B. & Ger, R. 1990 *Iterative functional equations: encyclopedia of mathematics and its applications*, vol. 32. Cambridge, UK: Cambridge University Press.
- Lifanov, I. K. 1996 *Singular integral equations and discrete vortices*. Amsterdam, The Netherlands: VSP.
- McPhedran, R. C. 1986 Transport properties of cylinder pairs and of the square array of cylinders. *Proc. R. Soc. Lond. A* **408**, 31–43. (doi:10.1098/rspa.1986.0108)
- McPhedran, R. C. & Milton, G. W. 1987 Transport properties touching of cylinder pairs and of the square array of touching cylinders. *Proc. R. Soc. Lond. A* **411**, 313–326. (doi:10.1098/rspa.1987.0069)
- McPhedran, R. C. & Nicorovici, N. A. 1997 Effective dielectric constant of arrays of elliptical cylinders (invited talk). In Proc. Fourth Int. Conf. on Electrical Transport and Optical Properties of Inhomogeneous Media (ETOPIM4). *Physica A* **241**, 173–178. (doi:10.1016/S0378-4371(97)00078-2)
- Milton, G. W. 2002 *The theory of composites*. Cambridge, UK: Cambridge University Press.
- Mityushev, V. 1993 Plane problem for the steady heat conduction of material with circular inclusions. *Arch. Mech.* **45**, 211–215.
- Mityushev, V. 1999 Transport properties of two-dimensional composite materials with circular inclusions. *Proc. R. Soc. Lond. A* **455**, 2513–2528. (doi:10.1098/rspa.1999.0414)
- Mityushev, V. 2001 Transport properties of doubly periodic array of circular cylinder and optimal design problem. *Appl. Math. Optim.* **44**, 17–31. (doi:10.1007/s00245-001-0013-y)
- Mityushev, V. 2005 R-linear problem on torus and its application to composites. *Complex Var.* **50**, 621–630. (doi:10.1080/02781070500086982)
- Mityushev, V. V. & Rogosin, S. V. 2000 *Constructive methods for linear and nonlinear boundary value problems for analytic functions theory*. Boca Raton, FL: Chapman & Hall/CRC.
- Mityushev, V. V., Pesetskaya, E. & Rogosin, S. V. 2008 Analytical methods for heat conduction in composites and porous media. In *Cellular and porous materials: thermal properties simulation and prediction* (eds A. Ochsner, G. E. Murch, M. J. S. de Lemos), pp. 121–164. Weinheim, Germany: Wiley-VCH Verlag GmbH & Co. KGaA.
- Movchan, A. B., Movchan, N. V. & Poulton, C. G. 2002 *Asymptotic models of fields in dilute and densely packed composites*. London, UK: Imperial College Press.
- Movchan, A. B., Milton, G. W. & Serkov, S. K. 2003 Realizable (average stress, average strain) pairs in a plate with holes. *SIAM J. Appl. Math.* **63**, 987–1028. (doi:10.1137/S0036139901395717)
- Nicorovici, N. A. & McPhedran, R. C. 1996 Transport properties of arrays of elliptical cylinders. *Phys. Rev. E* **54**, 1945–1957. (doi:10.1103/PhysRevE.54.1945)
- Rylko, N. 2000 Transport properties of a rectangular array of highly conducting cylinders. *J. Eng. Math.* **464**, 391–407.
- Yardley, J. G., McPhedran R. C. & Nicorovici, N. A. 1999 Addition formulas and the Rayleigh identity for arrays of elliptical cylinders. *Phys. Rev. E* **60**, 6068–6080. (doi:10.1103/PhysRevE.60.6068)
- Yardley, J. G., Reuben A. J. & McPhedran, R. C. 2001 The transport properties of layers of elliptical cylinders. *Proc. R. Soc. Lond. A* **457**, 395–423. (doi:10.1098/rspa.2000.0672)
- Zimmerman, R. W. 1996 Effective conductivity of a two-dimensional medium containing elliptical inhomogeneities. *Proc. R. Soc. Lond. A* **452**, 1713–1727. (doi:10.1098/rspa.1996.0091)