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ANALYTICAL METHODS FOR HEAT CONDUCTION IN COMPOSITES[1]

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Abstract. Analytical methods unifying the study of heat conduction in various type of composite materials are described. Analytical formulas for the effective (macroscopic) conductivity tensor are presented.

Key words: 2D composites, heat conduction, analytical methods, effective conductivity.

1 Introduction

The goal of this survey paper is to describe analytical methods applied to the study of steady heat conduction in various types of composites. We present several exact and approximate analytical formulas for the effective (macroscopic) conductivity tensor which are deduced by using different approaches based on the recent results in the theory of partial differential equations and

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complex analysis. The study of effective characteristics has recently become a separate subject with its own philosophy and machinery. The most popular models studied for composites are those of conductivity, elasticity, elastoplasticity and thermo-elasticity (see, e.g., [1, 5, 13, 18]). We restrict our attention on the conducting properties of the composite in the steady state situation. Moreover, so called two-dimensional composite materials are mainly on the discussion since in this case it is possible to deduce certain analytical formulas for the effective conductivity by using the technique of harmonic and analytic functions.

The analytical approach to the study of heat conduction allows us to unify partly the theory of the effective thermal properties in composite materials. This paper is connected with [17].

2 Mathematical Models for Heat Conduction in Composites

Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^N$, for $N = 1, 2, 3$. Let Ω be a domain occupied by the conducting material. We denote by $T(\mathbf{x})$ the temperature distribution, and by $\mathbf{q}(\mathbf{x})$ the heat flux.

The equations representing dependence of the flux $\mathbf{q}(\mathbf{x})$ on the temperature $T(\mathbf{x})$ are called (the heat conduction) *constitutive relation.* In the linear case it has a form of Fourier's law

$$\mathbf{q} = -\lambda \nabla T, \tag{2.1}$$

where ∇T is the gradient of $T(\mathbf{x})$.

In the linear case the proportionality coefficient λ depends solely on spatial variable $\mathbf{x}$, λ is called *the local thermal conductivity* or simply the conductivity. The thermal conductivity is considered as a scalar positive function $\lambda = \lambda(\mathbf{x})$ for locally isotropic materials and as a tensor function for locally anisotropic materials which in Cartesian coordinates has the form of the symmetric positively defined matrix

$$\lambda = \lambda(\mathbf{x}) = \begin{pmatrix} \lambda_{11}(\mathbf{x}) & \lambda_{21}(\mathbf{x}) & \lambda_{31}(\mathbf{x}) \\ \lambda_{12}(\mathbf{x}) & \lambda_{22}(\mathbf{x}) & \lambda_{23}(\mathbf{x}) \\ \lambda_{13}(\mathbf{x}) & \lambda_{23}(\mathbf{x}) & \lambda_{33}(\mathbf{x}) \end{pmatrix}.$$

If the conductivity λ depends on the temperature, i.e., $\lambda = \lambda(\mathbf{x}, T)$, this case is called the *non–linear* heat conduction.

At the presence of sources and sinks with intensity $f(\mathbf{x})$, we get the following relation

$$\nabla \cdot \mathbf{q} = f \quad in\ \Omega.$$

If there is no source or sink, then we have the free divergence equation

$$\nabla \cdot \mathbf{q} = 0. \tag{2.2}$$

Substituting (2.2) into (2.1) we obtain an elliptic equation

$$\nabla \cdot (\lambda \nabla T) = 0. \tag{2.3}$$

For macroscopically isotropic material the conductivity $\lambda(\mathbf{x})$ is a constant:

$$\nabla^2 T = 0. \tag{2.4}$$

3 Boundary Value Problems

Let us present different types of boundary value problems for heat conduction in composites (i.e., boundary value problems for equations (2.3) and (2.4)). To be more precise, we formulate these problems in the case of composites consisting of the matrix (which is a multiply connected region Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$) with outer boundary curve/surface Γ and of n inclusions D_k, $k = 1, \ldots, n$, encircled by smooth closed curves/surfaces $L_k = \partial D_k$. It is convenient to use the notation $\Lambda(\mathbf{x})$ for the conductivity tensor for the material occupied by the host domain Ω, and $\Lambda_k(\mathbf{x})$, $k = 1, \ldots, n$, – for the conductivity tensors for the material occupied by the corresponding inclusions.

We suppose that either $N = 2$ or $N = 3$, and make corresponding remarks when these situations differ essentially. According to the chosen orientation on the boundary of Ω, we will denote by $T(\mathbf{t})$ the boundary values of the temperature distribution on Γ, and the boundary limits on L_k of the temperature from the domain Ω and domains D_k by $T^+(\mathbf{t})$, $T_k^-(\mathbf{t})$, respectively.

The given temperature distribution $f(\mathbf{t})$ on the outer boundary Γ leads to the Dirichlet condition on Γ:

$$T(\mathbf{t}) = f(\mathbf{t}), \quad \mathbf{t} \in \Gamma. \tag{3.1}$$

If the outer boundary constitutes the ideal thermal isolator (i.e., there is no heat exchange between the composite and the medium outside of it), then we arrive at the homogeneous Neumann condition

$$\frac{\partial T}{\partial \mathbf{n}}(\mathbf{t}) = 0, \quad \mathbf{t} \in \Gamma. \tag{3.2}$$

If there is a heat transfer trough the outer boundary when the normal heat flux $\mathbf{q} \cdot \mathbf{n}$ is known at the outer surface, then condition (3.2) should be replaced by a more complicated one (see (2.1))

$$\lambda \nabla T \cdot \mathbf{n}(\mathbf{t}) = g(\mathbf{t}), \quad \mathbf{t} \in \Gamma. \tag{3.3}$$

Instead of (3.3) the heat transfer satisfying Newton's law can be considered at the boundary

$$\lambda \frac{\partial T}{\partial \mathbf{n}}(\mathbf{t}) + \gamma T(\mathbf{t}) = h(\mathbf{t}), \quad \mathbf{t} \in \Gamma. \tag{3.4}$$

It is also called the third type boundary value problem.

Other types of condition arise on internal components of the boundary of Ω, i.e., on the interface matrix-inclusions. The most natural are continuity of the temperature and of the heat flux. Then, they have the following form (perfect contact conditions):

$$T^+(\mathbf{t}) = T_k^-(\mathbf{t}), \ \lambda(\mathbf{x}) \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) = \lambda_k(\mathbf{x}) \frac{\partial T_k^-}{\partial \mathbf{n}}(\mathbf{t}), \ \mathbf{t} \in L_k, \ k = 1, \ldots, n. \tag{3.5}$$

It is also natural to assume that the temperature distribution and the normal heat flux have jumps along a part of the interface matrix-inclusions:

$$T^+(\mathbf{t}) - T_k^-(\mathbf{t}) = h_k(\mathbf{t}), \quad k = 1, \ldots, n, \tag{3.6}$$
$$\lambda(\mathbf{x}) \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) = \lambda_k(\mathbf{x}) \frac{\partial T_k^-}{\partial \mathbf{n}}(\mathbf{t}) + g_k(t).$$

If at least a part of the interface matrix-inclusions consists of poorly conducting material then we have to replace the first series of the above conditions by a more complicated one, namely, we have the following problem:

$$\lambda(\mathbf{x}) \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) + \gamma_k (T^+(\mathbf{t}) - T_k^-(\mathbf{t})) = 0, \tag{3.7}$$
$$\lambda(\mathbf{x}) \frac{\partial T^+}{\partial \mathbf{n}}(\mathbf{t}) = \lambda_k(\mathbf{x}) \frac{\partial T_k^-}{\partial \mathbf{n}}(\mathbf{t}), \quad \mathbf{t} \in L_k.$$

The coefficients γ_k^{-1} introduced in (3.7) are known as the Kapitza resistance (see, e.g., [13]). The limit cases $\gamma_k = 0$, and $\gamma_k = \infty$ were discussed in [16].

A special problem can be also considered, namely, with the boundary conditions given on the exterior boundary and the domains D_k occupied by an ideal conductor ($\lambda_k = \infty$). In this case, we arrive at the modified Dirichlet problem [12]

$$T(\mathbf{t}) = t_k, \quad \mathbf{t} \in L_k, \tag{3.8}$$

where t_k are undetermined constants which have to be found in the solution to the problem.

4 Complex Potentials

The aim of this subsection is to rewrite equations as well as boundary value problems for heat conduction in composites (or in porous media) in terms of complex analysis. Thus, we have studied here only the two-dimensional situation ($N = 2$) considering the corresponding domains as domains on the complex plane $\mathbb{C}$. In this case, it is supposed that the heat flux is spreading in a direction orthogonal to the cylinder in which parallel cylindrical inclusions are implemented. The base of the cylinder is a multiply connected domain Ω, and the bases of the inclusions are domains D_k. There is also another statement of the 2D problem when a thick plate with isolated sides is considered.

First, we consider the limit cases when Ω is occupied by a conducting material on the boundary of which boundary conditions (3.1), (3.3) or (3.4) are given. Consider the Dirichlet problem (3.1). It is known that each harmonic function in a simply connected domain is the real part of a complex potential. If a function $T(x, y)$ is harmonic in a multiply connected domain Ω then it can be expressed as:

$$T(z) = Re\, [\Phi(z) + \sum_{k=1}^{n} A_k \ln(z - z_k)], \quad z = x_1 + ix_2 \in \Omega, \tag{4.1}$$

according to the decomposition theorem (see [16]). Here, function $\Phi(z)$ is analytic and single-valued in Ω, and A_k are real numbers. If we assume that $\infty \in \Omega$, D_k $(k = 1, 2, \ldots, n)$ are connected components of the complement of Ω to $\mathbb{C}$, and z_k are points in D_k, then the connectivity of Ω is equal to $n - 1$ and

$$\sum_{k=1}^{n} A_k = 0.$$

Substituting $T(z)$ from (4.1) into (3.1), we arrive at the boundary value problem with respect to $\Phi(z)$. The constants A_k have also to be determined. One can find a discussion of this problem for multiply connected domains in [12] and a complete solution to this problem for any circular multiply connected domain is given in [16]. A similar argument can be applied to the problems (3.2) and (3.4).

Consider now the modified Dirichlet problem (3.8). In this case, instead of (4.1) we have $T(z) = Re\,\Phi(z)$. However, the undetermined constants t_k are included in the boundary condition

$$Re\,\Phi(t) = t_k, \quad t \in L_k \quad (k = 1, 2, \ldots, n).$$

We also suppose (again for simplicity) that the materials inside matrix and inclusions are isotropic and homogeneous, which means the constancy of conductivity coefficients $\lambda, \lambda_k,\ k = 1, \ldots, n$. Therefore, the temperature T is a harmonic function in the domains Ω and $D_k,\ k = 1, \ldots, n$ (*i.e.*, satisfies in these domains the Laplace equation (2.4)).

Let T, and T_k be temperature distributions in Ω and $D_k,\ k = 1, \ldots, n$, respectively, continuously differentiable up to the boundaries of these domains satisfying (2.4). Suppose that the perfect contact relations (3.5) are valid on each curve $L_k = \partial D_k,\ k = 1, \ldots, n$. Then, one can introduce functions

$$\varphi(z) = T(z) + iV(z),\ z \in \Omega,\ \varphi_k(z) = \frac{\lambda + \lambda_k}{2\lambda}\left(T_k(z) + iV_k(z)\right),\ z \in D_k, \quad (4.2)$$

which are analytic in Ω, D_k, respectively, continuously differentiable in the closures of the considered domains. In fact (see, e.g., [9]), the function $\varphi(z)$ is in general a multi-valued analytic function since Ω is a multiply connected domain. But in our case, $T(z)$ possesses (see [8]) a unique harmonic extension up to the function, harmonic in a simply connected domain $D = \Omega \bigcup_{k=1}^{n} L_k \bigcup_{k=1}^{n} D_k$ due to the first relations in (3.5). Therefore, due to the uniqueness of analytic continuation $\varphi(z)$ is a single-valued analytic function in Ω as the restriction of the corresponding function defined on D.

In order to represent the boundary conditions (3.5) in the complex form, we write the normal and tangent derivatives on a fixed curve L_k:

$$\frac{\partial}{\partial \mathbf{n}} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial \mathbf{s}} = -n_2 \frac{\partial}{\partial x_1} + n_1 \frac{\partial}{\partial x_2}, \quad (4.3)$$

where $\mathbf{n} = n_1 + in_2$, $\mathbf{s} = n_2 - in_1$, and $z = x_1 + ix_2$. By applying the second operator of (4.3) to the first condition of (3.5), after some calculations (see [17]) applied to the new complex potentials in the domains Ω and D_k, respectively,

$$\psi = \frac{\partial \varphi}{\partial z} = \frac{\partial T}{\partial x_1} - i\frac{\partial T}{\partial x_2}, \quad \psi_k = \frac{\lambda_k + \lambda}{2\lambda}\frac{\partial \varphi_k}{\partial z} = \frac{\lambda_k + \lambda}{2\lambda}\left(\frac{\partial T_k}{\partial x_1} - i\frac{\partial T_k}{\partial x_2}\right),$$

we arrive at the following conjugation condition

$$\psi^+(t) = \psi_k^-(t) + \rho_k \overline{\mathbf{n}^2 \psi_k^-(t)}, \quad t \in L_k, \tag{4.4}$$

where $\rho_k = \dfrac{\lambda_k - \lambda}{\lambda_k + \lambda}$ is a contrast parameter introduced by Bergman [2, 3]. Integrating (4.4) along L_k with constant of integration equal to zero (see [16]), we obtain the following boundary value problem for analytic functions in a multiply connected domain, namely, for the complex potentials φ, φ_k

$$\varphi^+(t) = \varphi_k^-(t) - \rho_k \overline{\varphi_k^-(t)}, \quad t \in L_k, \ k = 1, \ldots, n.$$

This problem is a special case of so-called $\mathbb{R}$-linear conjugation problem (Markushevich's problem) (see [4, 11] for the description of qualitative results on solvability of $\mathbb{R}$-linear conjugation problem with arbitrary coefficients).

If at least one of the first conditions in (3.5) is replaced by a non-zero jump condition, i.e., we have (3.6), then one can proceed in a similar way as before. We introduce the complex potentials by formulas (4.2). If h_k are smooth enough, e.g., $h_k \in \mathcal{C}^{1,\alpha}(L_k)$, then one can find (single-valued) analytic in D_k functions $h_k^-(z)$ satisfying the Schwarz boundary conditions:

$$\operatorname{Re} h_k^-(t) = h_k(t), \quad t \in L_k.$$

Then, the first conditions of (3.6) can be rewritten in the form

$$T^+(t) - \widetilde{T_k^-}(t) = 0, \quad t \in L_k, \quad k = 1, \ldots, n,$$

where $\widetilde{T_k^-}(z) = T_k^-(z) + \operatorname{Re} h_k^-(z)$, $z \in D_k$. The same result is valid for the second condition (3.6). As a result, we have the following boundary value problem with non-zero inhomogeneous term $c_k(t)$ on at least one curve L_k:

$$\varphi^+(t) = \varphi_k^-(t) - \rho_k \overline{\varphi_k^-(t)} + c_k(t), \quad t \in L_k.$$

Exact calculation of the inhomogeneous term can be easily done. It does not have much influence on further analysis.

Application of the same arguments to (3.7) yields a $\mathbb{R}$-linear conjugation problem with derivatives. Let us assume for simplicity that the inclusions are circular cylinders, *i.e.*, $D_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$, $k = 1, \ldots, n$. Then, problem (3.7) becomes

$$\varphi^+(t) = \varphi_k^-(t) - \rho_k \overline{\varphi_k^-(t)} + \mu_k (t - a_k)(\varphi_k^-)'(t) + \mu_k \frac{r_k^2}{t - a_k}\overline{(\varphi_k^-)'(t)},$$

where $|t - a_k| = r_k$, $\mu_k = \dfrac{1 + \rho_k}{2 r_k \gamma_k}$.

Finally, we have to rewrite the boundary conditions on the outer boundary Γ in the complex form too, e.g., the Dirichlet condition (3.1). By solving

$$\mathrm{Re}\, f_0(t) = f(t), t \in \Gamma,$$

with respect to the function $f_0(z)$ analytic outside $D = \Omega \bigcup_{k=1}^{n} L_k \bigcup_{k=1}^{n} D_k$, *i.e.*, in $ext\, D$. Then by introducing an auxiliary unknown function $\varphi_0(z)$ analytic in $ext\, D$, $\varphi_0(\infty) = 0$ and using the same complex potential $\varphi(z)$ for Ω, we rewrite (3.1) in the form of $\mathbb{R}$-linear conjugation problem:

$$\varphi^+(t) = \varphi_0(t) - \overline{\varphi_0(t)} + f_0(t), t \in \Gamma.$$

A similar approach is used for the Neumann problem (3.2) (for complex potential ψ) (see [16]).

5 Effective Conductivity Tensor

Although the notation *effective conductivity tensor* is intuitively clear for physicists and engineers, the rigorous mathematical definition of the effective conductivity tensor needs a certain theoretical justification. One of the possible ways for such a justification is the use of homogenization theory.

Following [1, 5, 10], consider a periodic composite. Let the linear sizes of the periods be of order $\varepsilon << L$, where L is the linear order of the sample D bounded by a simple closed curve Γ. Consider the Dirichlet problem [10] (p. 20 Russian ed.) in $H_0^1(D)$

$$\begin{cases} \nabla(\Lambda_\varepsilon(\mathbf{x})\nabla T_\varepsilon(\mathbf{x})) = 0, \\ T_\varepsilon(\mathbf{t}) = f(\mathbf{t}), \quad \mathbf{t} \in \Gamma. \end{cases} \tag{5.1}$$

Let

$$\Lambda_\varepsilon(\mathbf{x})\nabla T_\varepsilon(\mathbf{x}) \rightharpoonup \widehat{\Lambda}\nabla T_0 \ \text{ in } L_2(D), \tag{5.2}$$

where $\rightharpoonup$ means the weak convergence in $L_2(D)$, $\widehat{\Lambda}$ is a constant tensor, and T_0 is a solution of the Dirichlet problem

$$\nabla(\widehat{\Lambda}\nabla T_0(\mathbf{x})) = 0, \quad T_0(\mathbf{t}) = f(\mathbf{t}), \quad \mathbf{t} \in \Gamma.$$

Then, the tensor $\widehat{\Lambda}$ is called the effective conductivity tensor. The homogenization theory justifies the existence of the weak limit (5.2) and the independence of the limit of the shape of Γ and boundary conditions. For instance, instead of the Dirichlet condition (5.1), the Neumann condition can be taken. Moreover, the homogenization theory implies that $\widehat{\Lambda}$ can be calculated by the formula

$$\widehat{\Lambda}\mathbf{q} = \langle \Lambda(\mathbf{x})\nabla T(\mathbf{x})\rangle, \tag{5.3}$$

where $\langle F(\mathbf{x})\rangle$ denote the average of the magnitude F over the cell Q

$$\langle F(\mathbf{x})\rangle = \frac{1}{|Q|}\int_Q F(\mathbf{x})d\mathbf{x},$$

$|Q|$ is the area of Q. For the definition of the periodic cell and the method of its determination see [14]. The function $T(\mathbf{x})$ is a solution of the quasi-periodic problem:

$$\begin{aligned}
&\nabla(\Lambda(\mathbf{x})\nabla T(\mathbf{x})) = 0, \ \mathbf{x} \in Q, \\
&T(x_1+\alpha, x_2, x_3) - T(x_1, x_2, x_3) = q_1, \\
&T(x_1, x_2+\beta, x_3) - T(x_1, x_2, x_3) = q_2, \\
&T(x_1, x_2, x_3+\gamma) - T(x_1, x_2, x_3) = q_3.
\end{aligned} \tag{5.4}$$

Here, $\mathbf{q} = (q_1, q_2, q_3)$ is the external flux. One can see that $\widehat{\Lambda}$ is completely determined by (5.3) via solution to three problems (5.4) with $\mathbf{q} = (1, 0, 0)$, $\mathbf{q} = (0, 1, 0)$, $\mathbf{q} = (0, 0, 1)$.

In general, $\widehat{\Lambda}$ is a symmetric positively defined tensor. It can be reduced to the diagonal form:

$$\widehat{\Lambda} = \begin{pmatrix} \widehat{\lambda}_1 & 0 & 0 \\ 0 & \widehat{\lambda}_2 & 0 \\ 0 & 0 & \widehat{\lambda}_3 \end{pmatrix}. \tag{5.5}$$

More precisely, there exists a coordinate system in which the tensor $\widehat{\Lambda}$ has the diagonal form (5.5). The axes x'_j $(j = 1, 2, 3)$ of this new coordinate system are called *the principal axes.* The component $\widehat{\lambda_j}$ $(j = 1, 2, 3)$ is called the conductivity in the x'_j–direction. The tensor ellipsoid, invariants of the tensor and other fundamental properties of tensors can be found in standard textbooks on tensor algebra.

The tensor $\widehat{\Lambda}$ for macroscopically isotropic composites has the form:

$$\widehat{\Lambda} = \widehat{\lambda}\mathbf{I},$$

where $\mathbf{I}$ is the identity tensor, i.e., in this case $\widehat{\lambda} := \widehat{\lambda}_1 = \widehat{\lambda}_2 = \widehat{\lambda}_3$. The scalar $\widehat{\lambda}$ is called the effective conductivity.

The variational statement of the problem implies the formula:

$$\widehat{\lambda}\, \mathbf{q} = \inf_{u \in H^1_{per}(Q)} \langle \lambda(\mathbf{x}) |\nabla u(\mathbf{x})|^2 \rangle = \langle \lambda(\mathbf{x}) |\nabla T(\mathbf{x})|^2 \rangle.$$

Consider a 2D representative symmetric cell. Then, the periodicity cell problem is reduced to the mixed problem for the domain Q. In this case, the following formula can used for the effective conductivity in the x_1–direction:

$$\widehat{\lambda}_1 = \frac{4}{\alpha q_1} \int_{-\alpha/4}^{\alpha/4} \lambda\left(x_1, \frac{\beta}{4}\right) \frac{\partial T}{\partial x_1}\left(x_1, \frac{\beta}{4}\right) dx_1.$$

This formula expresses that the effective conductivity in the x_1–direction is equal to the average flux passing along the symmetry segment

$$x_2 = \frac{\beta}{4}, \quad -\frac{\alpha}{4} < x_1 < \frac{\alpha}{4}$$

divided by the jump of the temperature $q_1/2$ per the half–periodicity cell. Similar formulas take place for the conductivities $\widehat{\lambda}_2$ and for corresponding coefficients in $\mathbb{R}^3$ (i.e., for 3D-composites).

Consider now an application of the formula (5.3) to 2D matrix-inclusion composites. Using the functions $\psi^{(j)}(z)$ (partial linearly independent solutions of the problem (4.4), satisfying

$$\psi(z+\alpha)-\psi(z)=0, \quad \psi(z+i\alpha^{-1})-\psi(z)=0),$$

we obtain the components of $\widehat{\Lambda}$:

$$\widehat{\lambda}_{11}-i\widehat{\lambda}_{12}=1+2\sum_{k=1}^{n}\rho_k\int\limits_{D_k}\psi^{(1)}(z)\,dx_1dx_2,$$

$$\widehat{\lambda}_{22}+i\widehat{\lambda}_{12}=1+2i\sum_{k=1}^{n}\rho_k\int\limits_{D_k}\psi^{(2)}(z)\,dx_1dx_2.$$

For macroscopically isotropic composites, we have:

$$\widehat{\lambda}=1+2\sum_{k=1}^{n}\rho_k\int_{D_k}\psi^{(1)}(z)dx_1dx_2. \tag{5.6}$$

Consider the case when the inclusions D_k are disks $|z-a_k|<r_k$. Then, application of the mean value theorem to (5.6) yields:

$$\widehat{\lambda}=1+2\sum_{k=1}^{n}\rho_k\pi r_k^2\psi^{(1)}(a_k).$$

6 Review of Known Formulas

Consider a two-component macroscopically isotropic composite medium consisted of a collection of non-overlapping identical balls of conductivity λ_1 imbedded into a host medium of conductivity λ. The effective conductivity $\widehat{\lambda}$ of the considered inhomogeneous medium is calculated by the famous Clausius-Mossotti approximation (CMA)

$$\frac{\widehat{\lambda}}{\lambda}\approx\frac{1+2\beta\nu}{1-\beta\nu}, \tag{6.1}$$

where $\beta=\dfrac{\lambda_1-\lambda}{\lambda_1+2\lambda}$, ν is the concentration of the spheres. The formula (6.1) holds for dilute composites when the concentration ν is small.

In the 2D case, CMA becomes

$$\frac{\widehat{\lambda}}{\lambda}\approx\frac{1+\rho\nu}{1-\rho\nu}, \tag{6.2}$$

where $\rho = \dfrac{\lambda_1 - \lambda}{\lambda_1 + \lambda}$ is the 2D contrast parameter. Here ν is the area concentration of disks on the plane (the section of the fiber composite perpendicular to the direction of fibers).

The formulas (6.1) and (6.2) can be deduced in the framework of Maxwell's formalism which is based on solution to the problem for one inclusion. The same method can be applied to inclusions of other shapes.

Generalized Keller-Dykhne formula (self-dual two-phase system with arbitrary concentration ν, compact inclusions of one phase into another) (see also [7])

$$\lambda_e = {\lambda_1}^{\nu} {\lambda_2}^{1-\nu}.$$

In series of papers by Craster and Obnosov (see, e.g., [6, 7]), exact formulas for the effective conductivity tensor of the few-phases checkerboard composites have been deduced. The proof is based on the explicit representations of the local fields for various types of such composites.

In the case of doubly periodic four–phase checkerboard composite when the representative rectangle has the lengths of the sides l, h the local conductivity $\lambda = \lambda(\mathbf{x})$ takes the value λ_j in j–phase ($j = 1, 2, 3, 4$). By using the complete elliptic integral

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}},$$

where the parameter m is implicitly defined by $K(m)/K(1-m) = l/h$, and the parameters

$$\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \quad \sigma_2 = \lambda_1\lambda_3 - \lambda_2\lambda_4,$$

$$\sigma_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$$

one can introduce the function

$$k(m, \nu) = \frac{l}{h} \frac{P_{\frac{\nu}{2} - \frac{1}{2}}(2m - 1)}{P_{\frac{\nu}{2} - \frac{1}{2}}(1 - 2m)},$$

where P_μ is the Legendre function of the first kind. The effective conductivity of the considered composite is then explicitly given by the formulas

$$\hat{\lambda}_1 = \frac{1}{k(m, \nu)} \left[\frac{(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)}{(\lambda_2 + \lambda_3)(\lambda_4 + \lambda_1)} \right]^{\frac{1}{2}} \left(\frac{\sigma_1}{\sigma_3} \right)^{\frac{1}{2}},$$

$$\hat{\lambda}_2 = k(m, \nu) \left[\frac{(\lambda_2 + \lambda_3)(\lambda_4 + \lambda_1)}{(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)} \right]^{\frac{1}{2}} \left(\frac{\sigma_1}{\sigma_3} \right)^{\frac{1}{2}}.$$

The effective conductivity of the square array of the doubly periodic composite with one inclusion in the cell is determined in [15] by the following equality (there is also a proof that $\psi(0)/c$ is real)

$$\widehat{\lambda} = 1 + 2\rho\pi r^2 \frac{\psi(0)}{c}.$$

Here $\psi(z)$ is a solution to certain $\mathbb{R}$-linear boundary value problem which is solved by the method of the functional equation (see [16]). Representing this solution in term of Eisenstein's series yields the exact formula for effective conductivity

$$\widehat{\lambda} = 1 + 2\rho\pi r^2 \\ + 2\pi \sum_{k=1}^{\infty} \rho^{k+1} \sum_{n_1,n_2,\ldots n_k} \sigma_{n_1}^{(1)} \sigma_{n_2}^{(n_1)} \ldots \sigma_{n_{k-1}}^{(n_{k-2})} \sigma_1^{(n_k)} r^{4(n_1+n_2+\cdots+n_k)-2(k-1)},$$

where

$$\sigma_k^{(n)} = \frac{(2n+2k-3)!}{(2n-1)!(2k-2)!} S_{2(n+k-1)},$$

S_{2j} are Eisenstein-Rayleigh sums, and n_j run over unit to infinity in the sum.

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