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Exponent in one of the variables

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Abstract
A periodicity functional equation of one complex variable which characterizes the exponential function is discussed. This functional equation can be generalized to equation for functions depending on two complex variables. It is conjectured that the second functional equation also characterizes the exponent. Applications to representations of complex continuous elementary functions are discussed.

1. A functional equation

Let ζ denote a complex variables, $i = \sqrt{-1}$. A function $f(\zeta)$ is called entire, if it is analytic in the complex plane, i.e., $f(\zeta)$ is represented in the form of the absolutely convergent series for all $\zeta \in \mathbb{C}$

$$f(\zeta) = \sum_{k=0}^{\infty} f_k \zeta^k.$$

The functional equation

$$\varphi(\zeta + 2\pi i) = \varphi(\zeta) \tag{1}$$

in the class of entire functions has the general solution of the form

$$\varphi(\zeta) = \psi(\exp\zeta) = \sum_{k=0}^{\infty} \psi_k e^{\zeta k}, \tag{2}$$

where ψ is an arbitrary entire function. In order to prove (2) we consider equation (1) in the strip $D = \{z \in \mathbb{C} : 0 \leqslant Im\ z \leqslant 2\pi\}$. The conformal mapping $t = \exp z$ maps D onto $\mathbb{C}$ with cut along the positive half–axis. The functional equation (1) implies that the limit values of $\psi(t) = \varphi(\zeta)$ at the different edges of the cut coincide. Hence, $\psi(t)$ is an arbitrary function analytic in $\mathbb{C}$. This proves (2).

2. Exponent on the second variable

In the present section, we discuss a functional equation similar to (1). We state just a conjecture about solutions of new functional equation.

Consider a class $\mathcal{A}$ of functions entire in the variables z and w. Let φ satisfies the functional equation

$$\varphi(\zeta, \zeta + 2\pi i) = \varphi(\zeta, \zeta), \quad \zeta \in \mathbb{C}. \tag{3}$$

Conjecture 1. *φ is the exponential function with respect to the second variable w, i.e.,*

$$\varphi(\zeta, w) = h(\zeta, \exp w) = \sum_{k=0}^{\infty} h_k(\zeta) e^{wk} \tag{4}$$

for some $h \in \mathcal{A}$.

Remark 1. The function $\varphi(\zeta, w) = w$ does not satisfy (3). This function satisfies (4) with $h(\zeta, u) = \ln u$. One can see that h does not belong to $\mathcal{A}$, since it has a jump across the cut of the complex logarithm.

Remark 2. The function equation

$$\varphi(w, \zeta + 2\pi i) = \varphi(w, \zeta), \quad (\zeta, w) \in \mathbb{C}^2, \tag{5}$$

is equivalent to equation (1), but it is not equivalent to equation (3), since only (5) $\Longrightarrow$ (3) by substitution of $w = \zeta$ in (5).

3. Elementary functions

The functional equation (3) has applications to topologically non–elementary functions introduced by Arnold [1, 2].

For brevity we denote rational functions by Q. The functions Q, exp, log, *sin*, *cos*, *tg*, *ctg*, *arcsin*, *arccos*, *arctg*, *arcctg* are called by the basic elementary functions. Real elementary functions are such functions which can be built from a finite number of the basic elementary functions through composition and combinations using the four elementary operations. Usually, the radical x^α is referred to the basic elementary functions. However, it can be expressed through the above functions $x^\alpha = \exp(\alpha \log x)$.

All complex elementary functions can be built by the basic functions Q, exp, log, since all other basic in the real case functions are express through these functions by the formulas

$$sin\ z = \frac{1}{2i}(e^{iz} - e^{-iz}), \quad cos\ z = \frac{1}{2}(e^{iz} + e^{-iz}),$$

$$arcsin\ z == -i \log[iz + \sqrt{1-z^2}], \quad arctg\ z = \frac{i}{2}[\log(1-iz) - \log(1+iz)],$$

and so forth.

As it is noted before, log is a discontinuous function. Hence, if we wish to discuss only complex continuous elementary functions, we can take just Q, exp as the basic functions. But it is only a conjecture which can be formulated more precisely as follows.

Conjecture 2. *All complex continuous elementary functions can be built by the basic functions* Q, exp. *More precisely, any complex continuous elementary function (even it contains terms of the type* $\exp \circ \log$*) can be generated from a finite number of the basic functions* Q, exp.

Conjecture 2 is related to Conjecture 1, if we do not care about singularities in Conjecture 1. Let a complex continuous elementary function $f(z)$ contains $\log z$, hence it has the form $f(z) = F(z, \log z)$. Introduce the variable $\zeta = \log z$. Then $f(e^\zeta) = F(e^\zeta, \zeta)$. The function $F(e^w, \zeta)$ satisfies the functional equation (3). If Conjecture 1 is true, $F(e^w, \zeta)$ is an exponential function in the variable ζ (in the first variable it is already exponential). This implies that log in $F(z, \log z)$ disappears after simplifications.

References

[1] V. I. Arnold, Arnold's Problems, Springer and Phasis, 2005, p. 168-170.

[2] V. Mityushev, The $\wp$–function of Weierstrass is not topologically conjugated to exponent, Proc. Int. Conf. "Education, science and economics at universities. Integration to international educational area", Plock, Poland 2006