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Exact solution of the R–linear problem for a disk in a class of doubly periodic functions

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Abstract

The **R**–linear conjugation problem for a disk in the class of doubly periodic functions, i.e., the **R**–linear problem on the torus, is solved in the form of series by Eisenstein's functions. The result is applied to the calculation of the effective conductivity of composites with circular inclusions.

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1 Introduction

Let D_k be mutually disjoint simply connected domains in the complex plane $\mathbf{C}$ bounded by smooth curves ∂D_k $(k = 1, 2, \ldots, n)$, D be the complement of all the closures of ∂D_k with respect to the extended complex plane $\mathbf{C} \cup \{\infty\}$. Let ∂D_k be oriented in counter clockwise direction. Let $a(t)$, $b(t)$ and $c(t)$ be given Hölder continuous functions on $\partial D = - \cup_{k=1}^{n} \partial D_k$; $a(t) \neq 0$.

$\mathbf{R}$–linear conjugation problem: Find a function $\varphi(z)$ analytic in $D, D_1, \ldots, D_n$, continuous in the closures of the considered domains with the following conjugation condition

$$\varphi^{+}(t) = a(t)\varphi^{-}(t) + b(t)\overline{\varphi^{-}(t)} + c(t), \quad t \in \partial D. \tag{1}$$

In the case $b(t) = 0$ we arrive at the $\mathbf{C}$–linear conjugation problem [9].

Noether's theory for problem (1) has been constructed by Mikhajlov [14] by reducing it to a singular integral equation. Litvinchuk & Spitkovskii [11] studied the problem (1) for a circle by its reduction to a two-dimensional $\mathbf{C}$–linear problem with a certain matrix coefficient.

This paper is devoted to constructive solution to the problem (1). First, we point to the case $a(t) \equiv b(t)$ in which Mikhajlov [14] has solved this problem by reduction it to the Riemann–Hilbert problem.

Special cases of problem (1) when it is possible to construct its solution are selected in [15], for instance, the $\mathbf{R}$–linear conjugation problem with constant coefficients and circular inclusions ∂D_k. In this case (1) can be written in the form

$$\varphi^{-}(t) = \varphi^{+}(t) - \rho_k\overline{\varphi^{+}(t)} + c(t), \quad t \in \partial D_k,\ k = 1, 2, \ldots, n, \tag{2}$$

where ρ_k are constants. Frequently (2) is written in the following form [15]

$$\psi^{-}(t) = \psi^{+}(t) + \rho_k \left(\overline{n(t)}^{2}\right) \overline{\psi^{+}(t)} + c'(t), \quad t \in \partial D_k, \tag{3}$$

where $n(t)$ is the unit normal vector to ∂D_k written in the form of complex value, $\psi(z) = \varphi'(z)$. The $\mathbf{R}$–linear problem (2) with $-1 \leq \rho_k \leq 1$ corresponds to the problem of perfect contact for composite materials with inclusions occupying the domains D_k. It has been solved in [15] by the method of functional equations for circular inclusions D_k.

Recently, Craster & Obnosov in a series of brilliant papers [5]–[8] have solved the problem (3) for contours forming checkerboard periodic structures on the plane.

As it follows from the homogenization theory [1] in order to determine the effective properties of the composites one has to consider the so-called cell periodicity problem. In the case of the conductivity of two-dimensional materials we arrive at the **R**–linear problem (3) in a class of doubly periodic functions. There are two interpretations of the latter problem. First, one can consider it as an **R**–linear problem on a torus represented by a rectangle with glued opposite sides [9, 26]. On the other hand this problem can be stated as a problem for infinitely connected domains [9, 24]. In the present paper we apply the first approach. The homogeneous **R**–linear problem for a disk on torus has been solved by reducing it to a functional equation. Following [16] we use Eisenstein's series to construct exact solution of the functional equation. This solution yields an exact formula for the effective conductivity of the square array of disks.

2 R–linear problem on a torus

Let $\mathbf{M} \cong \mathbf{Z}^2$ be the set of complex numbers with integer real and imaginary parts. Consider a square lattice $\mathcal{Q}$ which is defined by two fundamental translation vectors expressed by the complex numbers 1 and i on the complex plane $\mathbf{C}$. Introduce the zero-th cell $Q_0 := \{z = t_1 + it_2 : -1/2 < t_j < 1/2 \ (j = 1, 2)\}$. The lattice $\mathcal{Q}$ consists of the cell $Q_m := \{z \in \mathbf{C} : z - m \in Q_0\}$, where $m \in \mathbf{M}$. Let the disk $D_1 = \{z \in \mathbf{C} : |z| < r\}$ lie in the cell Q_0, D be the complement of the closure of D_1 to Q_0. To find a function $\psi(z)$ analytic in D, D_1, continuous in the closure of the considered domains with the following conjugation condition

$$\psi^-(t) = \psi^+(t) + \rho \left(\frac{r}{t}\right)^2 \overline{\psi^+(t)}, \quad |t| = r, \tag{4}$$

where $\psi(z)$ is doubly periodic with respect to $\mathcal{Q}$

$$\psi(z+1) = \psi(z) = \psi(z+i). \tag{5}$$

The equality (3) is reduced to (4) for $|t| = r$ and $c'(t) = 0$. The given constant ρ satisfies the inequality $-1 < \rho < 1$. The problem (4)–(5) can be considered as the homogeneous **R**–linear problem for the unit circle on the torus represented by the cell Q_0. It can be also considered as an **R**–linear problem for the infinitely connected domain bounded by $|t - m| = r$ ($m = m_1 + im_2 \in \mathbf{M}$, i.e., m_1 and m_2 are integers).

In mechanics, the problem (4)–(5) corresponds to a problem for a composite material, when the conductivity of the matrix is normalized by unity and $\lambda_1 = \frac{1+\rho}{1-\rho}$ is the respective conductivity of the inclusions.

3 Classical Eisenstein-Rayleigh sums and Eisenstein series for the square lattice

In the present section we introduce the fundamental constants and functions of the elliptic function theory following Weil [25] and Akhiezer [2].

The Eisenstein summation method is defined as follows

$$\sum_{m_1,m_2} = \lim_{N\to\infty} \sum_{m_2=-N}^{m_2=N} \left(\lim_{M\to\infty} \sum_{m_1=-M}^{m_1=M} \right). \tag{6}$$

Using this summation we introduce the conditionally convergent sum

$$S_2 := \sum_{m_1,m_2}{}' \, (m_1 + im_2)^{-2} = \sum_{m}{}' \, m^{-2}, \tag{7}$$

where m_1 and m_2 run over all integer numbers except the pair $m_1 = m_2 = 0$. It is known [18] that $S_2 = \pi$. Following Eisenstein and Rayleigh we introduce the absolutely convergent sums

$$S_n := \sum_{m}{}' \, m^{-n}, \; n = 3, 4, \dots\,. \tag{8}$$

It is known that $S_n = 0$ for odd n. For even n an efficient algorithm has been proposed in [17] to calculate (8).

The Eisenstein series are defined as follows

$$E_n(z) := \sum_{m} (z - m)^{-n}\,, \; n = 2, 3, \dots\,. \tag{9}$$

The Eisenstein summation method (6) is applied to $E_2(z)$. The series $E_n(z)$ for $n = 3, 4, \dots$ as a function in z converge absolutely and almost uniformly in the domain $\mathbf{C} \backslash \mathbf{M}$. Each of the functions (9) is doubly periodic and has a pole of order n at $z = 0$.

The Eisenstein series and the Weierstrass function $\wp(z)$ are related by the identities

$$
\begin{aligned}
E_2(z) &= \wp(z)+\pi, && (10)\\
E_n(z) &= \frac{(-1)^n}{(n-1)!}\frac{d^{n-2}\wp(z)}{dz^{n-2}}, \quad n=3,4,\ldots && (11)
\end{aligned}
$$

The Eisenstein functions of the even order $E_{2n}(z)$ can be presented in the form of the series

$$E_{2n}(z)=\frac{1}{z^{2n}}+\sum_{k=0}^{\infty}\sigma_k^{(n)}z^{2(k-1)}, \tag{12}$$

where

$$\sigma_k^{(n)}=\frac{(2n+2k-3)!}{(2n-1)!(2k-2)!}S_{2(n+k-1)}. \tag{13}$$

4 Reduction of the R–linear problem to a functional equation

In the present section we reduce the **R**–linear problem (4)–(5) to a functional equation. At first, we introduce the operator

$$T_m\psi(z):=\left(\frac{r}{z-m}\right)^2\left(\overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)}-\overline{\psi(0)}\right), \tag{14}$$

where $m\in\mathbf{M}$.

Introduce the Banach space C_A of functions continuous in $|z|\le r$ and analytic in $|z|<r$ with the norm $||\psi(z)||=\max_{|t|=r}|\psi(z)|$.
Theorem [20] *Let $\sum_m$ and $\sum_m'$ denote Eisenstein's summation, respectively with the term $m=0$ and without it.*

(i) The series

$$\Psi_0(z)=\sum_m{}' \, T_m\psi(z)$$

converges absolutely and uniformly in the closure of the cell Q_0 for each $\psi\in C_A$ to a function analytic in Q_0 and continuous in its closure.

(ii) The series

$$\Psi(z)=\sum_m T_m\psi(z) \tag{15}$$

converges absolutely and uniformly in each compact subset of D to a function analytic in D continuous in its closure and doubly periodic with respect to the considered lattice.

(iii) *The linear operator*

$$T = {\sum_m}' \, T_m \tag{16}$$

is compact in C_A.

Using Eisenstein's summation one can rewrite (15) in the form

$$\Psi(z) = \sum_m \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} - \overline{\psi(0)} r^2 E_2(z), \tag{17}$$

since the sum

$$\Psi_e(z) = \sum_m \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} \tag{18}$$

is correctly defined by Eisenstein's summation. It is commutative with integrals, it can be differentiated term by term as an absolutely and uniformly convergent series (15). However, it is forbidden to change the order of summation in (18).

We present the unknown function $\psi(z)$ in the form of its Taylor expansion

$$\psi(z) = \sum_{k=0}^{\infty} \psi_k z^k. \tag{19}$$

Then

$$\overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} = \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} \frac{1}{(z-m)^k} \tag{20}$$

for each m. Substitution of (20) in (18) yields

$$\Psi_e(z) = \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} E_{k+2}(z). \tag{21}$$

Introduce the function

$$\Phi(z) = \begin{cases} \psi(z) - \rho \sum\limits_{k=0}^{\infty} \overline{\psi_k} r^{2k} (E_{k+2}(z) - z^{-k-2}), \ |z| \le r, \\[2ex] \psi(z) - \rho \sum\limits_{k=0}^{\infty} \overline{\psi_k} r^{2k} E_{k+2}(z), \ z \in D, \end{cases} \tag{22}$$

analytic in D and in the disk $|z| < r$. Calculate the jump of $\Phi(z)$ across $|t| = r$

$$\Delta = \Phi^+(t) - \Phi^-(t) = \psi^+(t) - \psi^-(t) - \rho \left(\frac{r}{t}\right)^2 \overline{\psi^-(t)}. \qquad (23)$$

Using (4) one can see that $\Delta = 0$. Applying the principle of analytic continuation and Liouville's theorem we obtain that $\Phi(z)$ is a constant, say c. This complex constant corresponds to the vector of the external flux applied to the composite [19]. Then the definition of $\Phi(z)$ in $|z| \le r$ yields the following functional equation with respect to $\psi \in C_A$

$$\psi(z) = \rho \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} (E_{k+2}(z) - z^{-k-2}) + c,\ |z| \le r, \qquad (24)$$

which can be also written in the form

$$\psi(z) = \rho \sum_m{}' \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} + c,\ |z| \le r. \qquad (25)$$

We have

$$\overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} = \overline{\psi(z^* - m)},$$

where $z^* = \frac{r^2}{\overline{(z-m)}} + m$ is the inversion with respect to the circle $|z - m| = r$. The inversion z^* transforms the disk $|z| \le r$ onto a closed disk D^* from $|z - m| < r$ when $m \ne 0$. The translation $z \mapsto z - m$ returns D^* to $|z| < r$. Therefore, the shift $\frac{r^2}{\overline{z-m}}$ maps $|z| \le r$ onto a closed disk lying in $|z| < r$. Equations with shifts into the domain are called iterative functional equations [10].

Theorem [20] *Equation* (24) *with* $|\rho| < 1$ *has a unique solution in* C_A. *This solution can be found by the method of successive approximations converging in* C_A, *i.e., uniformly convergent in* $|z| \le r$.

5 Solution to the functional equation

In the previous section we have noted that it is possible to apply the method of successive approximations to the functional equation (24) and hence to construct an approximate solution in symbolic form as it was done in [4, 19, 23]. However, in this case of one inclusion in the periodicity cell it is possible also to write explicitly each term of this approximation, i.e., it is possible to write the exact solution of the functional equation in form of a series with explicitly given terms. It is convenient to perform it using the functional equation (25).

We are looking for $\psi(z)$ in the form of the series

$$\psi(z) = c\sum_{k=1}^{\infty} \left(\rho r^2\right)^k \psi_k(z). \tag{26}$$

Then (25) yields the following recurrence relations

$$\psi_0(z) = 1,$$
$$\psi_k(z) = \sum_m{}' \frac{1}{(z-m)^2}\overline{\psi_{k-1}\left(\frac{r^2}{\overline{z-m}}\right)},\ k = 1, 2, \dots . \tag{27}$$

Applying (27) we obtain

$$\psi_k(z) = \sum_{\nu_1}{}' \sum_{\nu_2}{}' \cdots \sum_{\nu_k}{}' (\nu_1 - z)^{-2} \left(\overline{\nu_2} - \frac{r^2}{\nu_1 - z}\right)^{-2}$$
$$\times \left(\nu_3 - \frac{r^2}{\overline{\nu_2} - \frac{r^2}{\nu_1 - z}}\right)^{-2} \dots \left(\overline{\nu_k} - \frac{r^2}{\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}}\right)^{-2}, \tag{28}$$

where ν_l corresponds to m from (27). For definiteness (28) is written for even k.

We apply (9), (12) and (13) to the latest term from (28)

$$\sum_{\nu_k}{}' \left(\overline{\nu_k} - \frac{r^2}{\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}}\right)^{-2} = \sum_{n_1=1}^{\infty} \sigma_{n_1}^{(1)} \frac{r^{4(n_1-1)}}{\left(\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}\right)^{2(n_1-1)}}. \tag{29}$$

Substitution of (29) in (28) yields

$$\psi_k(z) = \sum_{n_1=1}^{\infty} \sigma_{n_1}^{(1)} r^{4(n_1-1)} \sum_{\nu_1, \nu_2, \dots, \nu_k}{}' (\nu_1 - z)^{-2} \left(\overline{\nu_2} - \frac{r^2}{\nu_1 - z}\right)^{-2}$$
$$\times \left(\nu_3 - \frac{r^2}{\overline{\nu_2} - \frac{r^2}{\nu_1 - z}}\right)^{-2} \dots \left(\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}\right)^{-2n_1}, \tag{30}$$

We now apply (9), (12) and (13) to the latest term from (30)

$$\begin{aligned}
&\sum_{\nu_{k-1}}{}' \left(\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\nu_{k-3} - \frac{r^2}{\cdots}}} \right)^{-2n_1} \\
&= \sum_{n_2=1}^{\infty} \sigma_{n_2}^{(n_1)} \frac{r^{4(n_2-1)}}{\left(\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\cdots}} \right)^{2(n_2-1)}}.
\end{aligned} \tag{31}$$

Then (30) becomes

$$\psi_k(z) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sigma_{n_1}^{(1)} \sigma_{n_2}^{(n_1)} r^{4(n_1+n_2-1)} \ldots$$

$$\times \sum_{\nu_1,\nu_2,\ldots,\nu_{k-2}}{}' (\nu_1 - z)^{-2} \left(\overline{\nu_2} - \frac{r^2}{\nu_1 - z} \right)^{-2} \cdots \left(\overline{\nu_{k-2}} - \frac{r^2}{\nu_{k-3} - \frac{r^2}{\cdots}} \right)^{-2n_2}. \tag{32}$$

We again apply (9), (12) and (13) to (32) and so forth. At the end we obtain the desired formula

$$\begin{aligned}
\psi(z) \quad = \quad & c \sum_{k=0}^{\infty} (\rho r^2)^k \sum_{n_1,n_2,\ldots,n_k} \sigma_{n_1}^{(1)} \sigma_{n_2}^{(n_1)} \ldots \sigma_{n_{k-1}}^{(n_{k-2})} \\
& \times E_{n_k}(z) r^{4(n_1+n_2+\cdots+n_k-k)}.
\end{aligned} \tag{33}$$

Here we have used (26).

The effective conductivity of the square array is determined by the following equality [19] (there is also a proof that $\psi(0)/c$ is real)

$$\widehat{\lambda} = 1 + 2\rho\pi r^2 \frac{\psi(0)}{c}. \tag{34}$$

Substitution of (26) and (33) to (34) yields the exact formula

$$\widehat{\lambda} = 1 + 2\pi \sum_{k=0}^{\infty} \rho^{k+1} \sum_{n_1,n_2,\ldots,n_k} \sigma_{n_1}^{(1)} \sigma_{n_2}^{(n_1)} \ldots \sigma_{1}^{(n_k)} r^{4(n_1+n_2+\cdots+n_k)-2(k-1)}. \tag{35}$$

6 Conclusion

The formula (35) can be considered as an expansion of $\widehat{\lambda}$ on the concentration of the inclusions πr^2 and the contrast parameter ρ. This analiticity is consistent with the previous general result of Bergman [3]. The formula (35) includes all known formulas for $\widehat{\lambda}$ approximated by πr^2 and ρ. However, in the case when $r \to 1/2$ and $\rho \to 1$ the series (35) diverges to $+\infty$. Asymptotic formulas for $\widehat{\lambda}$ in this case were obtained by McPhedran et al. [12] (see also papers cited therein). Direct application of (35) to this limit case is doubtful. One can find some notes on application of the functional equations to this case in [21].

It could be interesting to estimate $\psi(0)$ without direct solution to the problem (4) or to the functional equation (24) in the limit case. The general theory of bounds for $\widehat{\lambda}$ is presented by Milton [13] without an address to the **R**–linear problem.

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