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Representative cell in mechanics of composites and generalized Eisenstein–Rayleigh sums†

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Consider a two-dimensional two-component periodic composite made from a collection of non-overlapping, identical, circular disks, embedded in a matrix. The effective conductivity tensor can be written in the form of expansion on 'basic elements' which depend only on locations of the disks. These elements are expressed in terms of the Eisenstein series. The representative cell of a composite is defined as the minimal size periodicity cell corresponding to the set of basic elements calculated for the composite. An algorithm to determine the representative cell for a given composite is constructed.

Keywords: Representative element; Representative volume element; Eisenstein series; Boundary value problem; Effective conductivity

Mathematics Subject Classifications: 33E05; 74Q15

1. Introduction

One of the most important notations of composite materials is the *representative volume element* (RVE). One can give a vague physical definition of this term as follows. The RVE is a part of the material which is small enough from a macroscopical point of view and can thus be treated as a typical element of the heterogeneous medium. On the other hand, it is sufficiently large in the microscopical scale, and it represents a typical microstructure of the material under consideration. In the present article first we give a rigorous definition of the representative element. Then, we determine its minimal size. The geometrical interpretation of the problem is shown in figures 1 and 2. The large cell $Q'_{(0,0)}$ presented in figure 1 is replaced by a smaller one, $Q_{(0,0)}$ (see figure 2) with three inclusions per periodicity cell.

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†Dedicated to Professor Guochun Wen on the occasion of his 75th birthday.

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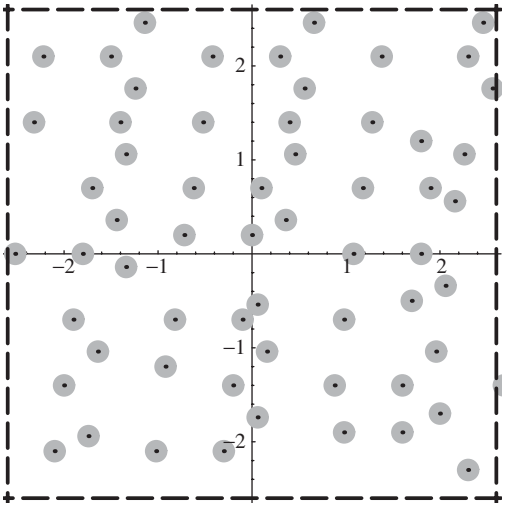

Figure 1. Large cell $Q'_{(0,0)}$ with 60 inclusions.

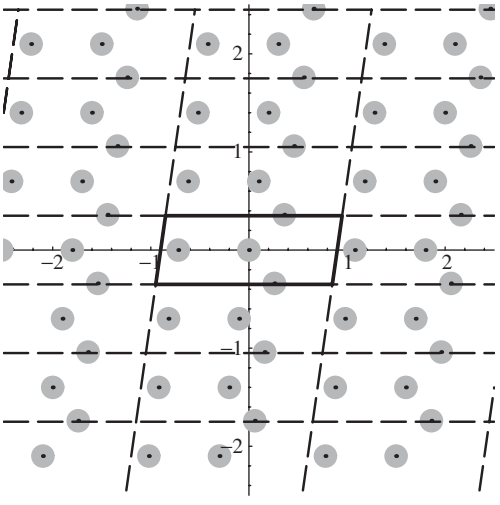

Figure 2. Small cell $Q_{(0,0)}$ with three inclusions centered at $a_1 = -0.92$, $a_2 = -0.36 + 0.36i$, $a_3 = 0$. It represents the large cell from the previous picture.

Drugan and Willis [1] estimated the minimal RVE size of the composite with uniformly distributed spherical inclusions considering the ensemble average of stress and strain 'at finite' and 'at infinite length'. The infinite length corresponds to the overall elastic constants of the homogenized materials. The RVE was introduced as a finite element, the mean constants of which are numerically closed to the mean constants at the infinite length. The authors derived quantitative estimates for the minimal RVE size in the case when matrix is reinforced by a random dispersion of non-overlapping identical spheres. Gusev [2] numerically calculated the overall elastic constants of individual Monte Carlo realizations with 8, 27 and 64 spheres and established that the scatter among the individual elastic constants is very small for the same uniform distribution of spheres. Adler [3] discussed questions of the

reconstruction of porous media by statistical data and numerically constructed RVE. Kolodziej [4] systematically applied the collocation method to compute effective conductivity of various two-dimensional composites.

In this study we consider two-dimensional two-component periodic composite medium made from a collection of non-overlapping, identical, circular disks, embedded in an otherwise uniform matrix. We discuss fields governed by the Laplace equation, when the inclusions have scalar conductivity λ and separated by a matrix of unit conductivity. Let $\rho = (\lambda - 1)/(\lambda + 1)$ be the contrast parameter introduced by Bergman [5]. Mityushev [6], and Berlyand and Mityushev [7] established that the effective conductivity tensor Λ_e of the composites considered has the form of a double series on the concentration of inclusions and on 'basic elements' which depend only on the locations of the inclusions. These basic elements are written in terms of the Eisenstein series [8]. Coefficients in the double series depend on ρ. We say that two composites are equivalent if expansions of their Λ_e have the same basic elements. Therefore, we divide the set of the composites with circular identical inclusions into classes of equivalence determined only by geometrical structure of the composite. In particular, composites with the same locations of inclusions but with different ρ belong to the same class of equivalence. Note that composites belonging to a class of equivalence can have different Λ_e; and composites from different classes can have the same Λ_e. Each composite material is represented by a periodicity cell. In each class of equivalence we choose a composite having the minimal size cell. This cell is called the representative cell of the considered class of equivalent composite materials.

We propose a constructive algorithm to determine the representative cell for any distribution of inclusions using only pure geometrical parameters. More precisely, at the beginning we calculate the *generalized Eisenstein–Rayleigh sums* depending on the centers of circular inclusions for a given large cell. Then using these sums we construct the (minimal) representative cell, i.e., we calculate its fundamental translation vectors and determine the positions of inclusions within this cell.

We use the elliptic (doubly periodic meromorphic) functions in the form of the *Eisenstein series* introduced by Eisenstein in 1847 and developed by Weil [8]. The classical lattice sums (the Eisenstein sums) were applied to the calculation of the effective conductivity tensor by Rayleigh [9] when a representative cell contains one inclusion. In section 2 we recall the classical Eisenstein–Rayleigh sums, the Eisenstein series and introduce the generalized Eisenstein–Rayleigh sums. The latter sums can be considered as a generalization of the classical sums to a cell with few inclusions. Section 3 is devoted to the effective conductivity tensor represented by a cell with few identical circular disks. In section 4 we describe a method on how to replace a periodic composite by another one with the same effective properties but with smaller size of the periodicity cell. Numerical examples illustrating the general theory are presented in subsections 4.2 and 4.3.

2. Generalized Eisenstein–Rayleigh sums

2.1. *Classical Eisenstein–Rayleigh sums*

In the present subsection, we introduce the fundamental parameters of the elliptic function theory following Weil [8] and Akhiezer [10]. Consider a lattice $\mathcal{Q}$ which is defined

by two fundamental translation vectors expressed by complex numbers ω_1 and ω_2 on the complex plane $\mathbb{C}$. For the definiteness we assume that $\operatorname{Im}\tau > 0$, where $\tau = \omega_2/\omega_1$. Introduce the $(0,0)$-cell $Q_{(0,0)} := \{z = t_1\omega_1 + t_2\omega_2 : -1/2 < t_j < 1/2\ (j = 1,2)\}$. The lattice $\mathcal{Q}$ consists of the cell $Q_{(m_1,m_2)} := \{z \in \mathbb{C} : z - m_1\omega_1 - m_2\omega_2 \in Q_{(0,0)}\}$, where m_1 and m_2 run over integer numbers.

The Eisenstein summation method is defined as follows:

$$\sum_{m_1,m_2} = \lim_{N\to\infty} \sum_{m_2=-N}^{m_2=N} \left(\lim_{M\to\infty} \sum_{m_1=-M}^{m_1=M} \right). \tag{1}$$

Using this summation, we introduce the conditionally convergent sum

$$S_2(\omega_1,\omega_2) := \sum_{m_1,m_2}{}' (m_1\omega_1 + m_2\omega_2)^{-2}, \tag{2}$$

where m_1 and m_2 run over all integer numbers except the pair $m_1 = m_2 = 0$. The sum (2) is slowly convergent. An efficient computations formula for $S_2(\omega_1,\omega_2)$ was deduced in [11]

$$S_2(\omega_1,\omega_2) = \frac{2}{\omega_1}\zeta\left(\frac{\omega_1}{2}\right), \tag{3}$$

where $\zeta(z)$ is the Weierstrass ζ-function. Rylko [12] deduced another efficient formula

$$S_2(\omega_1,\omega_2) = \left(\frac{\pi}{\omega_1}\right)^2 \left(\frac{1}{3} - 8\sum_{m=1}^{\infty} \frac{mh^{2m}}{1-h^{2m}}\right), \quad \text{where } h = \exp(\pi i\tau). \tag{4}$$

Following Eisenstein and Rayleigh, we introduce the absolutely convergent sums

$$S_n(\omega_1,\omega_2) := \sum_{m_1,m_2}{}' (m_1\omega_1 + m_2\omega_2)^{-n}, \quad n = 3,4,\ldots. \tag{5}$$

It is known that $S_n(\omega_1,\omega_2) = 0$ for odd n. For even n the Eisenstein–Rayleigh sums (5) can easily be calculated through the rapidly convergent infinite sums:

$$g_2(\omega_1,\omega_2) := \left(\frac{\pi}{\omega_1}\right)^4 \left(\frac{4}{3} + 320\sum_{m=1}^{\infty} \frac{m^3h^{2m}}{1-h^{2m}}\right), \tag{6}$$

$$g_3(\omega_1,\omega_2) := \left(\frac{\pi}{\omega_1}\right)^6 \left(\frac{8}{27} - \frac{448}{3}\sum_{m=1}^{\infty} \frac{m^5h^{2m}}{1-h^{2m}}\right). \tag{7}$$

Then

$$S_4(\omega_1,\omega_2) = \frac{g_2(\omega_1,\omega_2)}{60}, \quad S_6(\omega_1,\omega_2) = \frac{g_3(\omega_1,\omega_2)}{1400}. \tag{8}$$

The sums $S_{2n}(\omega_1, \omega_2)$ $(n \geq 4)$ are calculated by the recurrence formula

$$S_{2n}(\omega_1, \omega_2) = \frac{3}{(2n+1)(2n-1)(n-3)} \sum_{m=2}^{n-2} (2m-1)(2n-2m-1) S_{2m} S_{2(n-m)}. \tag{9}$$

2.2. *Eisenstein series*

In the present subsection we summarize the main facts of the Eisenstein series theory following Weil [8]. The Eisenstein series are defined as follows:

$$E_n(z; \omega_1, \omega_2) := \sum_{m_1, m_2} (z - m_1\omega_1 - m_2\omega_2)^{-n}, \quad n = 2, 3, \ldots. \tag{10}$$

The Eisenstein summation method (1) is applied to $E_2(z; \omega_1, \omega_2)$. The series $E_n(z; \omega_1, \omega_2)$ for $n = 3, 4, \ldots$ as a function in z converge absolutely and almost uniformly in the domain $\mathbb{C} \setminus \cup_{m_1, m_2} (m_1\omega_1 + m_2\omega_2)$. Each of the functions (10) is doubly periodic and has a pole of order n at $z = 0$. However, further it will be convenient to define the value of $E_n(z; \omega_1, \omega_2)$ at the point zero as follows:

$$E_n(0; \omega_1, \omega_2) := S_n(\omega_1, \omega_2). \tag{11}$$

The Eisenstein series and the Weierstrass function $\mathcal{P}(z; \omega_1, \omega_2)$ are related by the identities

$$E_2(z; \omega_1, \omega_2) = \mathcal{P}(z; \omega_1, \omega_2) + S_2(\omega_1, \omega_2), \tag{12}$$

$$E_n(z; \omega_1, \omega_2) = \frac{(-1)^n}{(n-1)!} \frac{\mathrm{d}^{n-2}}{\mathrm{d}z^{n-2}} \mathcal{P}(z; \omega_1, \omega_2). \tag{13}$$

2.3. *Generalized Eisenstein–Rayleigh sums*

We now proceed to introduce one of the most important mathematical objects of this study, the generalized Eisenstein–Rayleigh sums. Consider a set of points a_k $(k = 1, 2, \ldots, N)$ in the cell $Q_{(0,0)}$. Let p be a natural number; k_s runs over 1 to N, $n_j = 2, 3, \ldots$. Let $\mathbf{C}$ be the operator of complex conjugation. The value

$$e_{n_1 \ldots n_p}(\omega_1, \omega_2) := \frac{1}{N^{p+1}} \sum_{k_0 k_1 \ldots k_p} E_{n_1}(a_{k_0} - a_{k_1}) \overline{E_{n_2}(a_{k_1} - a_{k_2})} \cdots \mathbf{C}^p E_{n_p}(a_{k_{p-1}} - a_{k_p}) \tag{14}$$

is called the generalized Eisenstein–Rayleigh sum. The parameters ω_1 and ω_2 are omitted in E_n.

For instance, for $p = 1$ (14) implies

$$e_n(\omega_1, \omega_2) := \frac{1}{N^2} \sum_{k_0, k_1} E_n(a_{k_0} - a_{k_1}). \tag{15}$$

According to (11) $e_n(\omega_1, \omega_2)$ becomes the classical Eisenstein–Rayleigh sum $S_n(\omega_1, \omega_2)$ in the case $N = 1$.

All sums and series introduced in this section are constructed basing on the fixed fundamental translation vectors ω_1 and ω_2. We are also interested in the normalized Eisenstein series

$$E_n(z; 1, \tau) := \sum_{m_1, m_2} (z - m_1 - m_2\tau)^{-n}, \quad n = 2, 3, \ldots. \tag{16}$$

We have the relations

$$E_n(z; \omega_1, \omega_2) = \omega_1^{-n} E_n\left(\frac{z}{\omega_1}; 1, \tau\right), \tag{17}$$

$$e_{n_1 \ldots n_p}(\omega_1, \omega_2) = \omega_1^{-2k} e_{n_1 \ldots n_p}(1, \tau), \tag{18}$$

where $2k := n_1 + \cdots + n_p$. Note that further we need only the even sums $n_1 + \cdots + n_p$.

3. Structure of the effective conductivity tensor

The results of this section are based on the papers [6,7]. Consider the cell $Q_{(0,0)}$ with N non-overlapping circular disks D_k of the radius r with the centers $a_k \in Q_{(0,0)}$ $(k = 1, 2, \ldots, N)$. Let D_0 be the complement of the closure of all disks D_k to $Q_{(0,0)}$. We study the conductivity of the doubly periodic composite material, when the domains $D_{\text{per}} := \cup_{(m_1, m_2)}(D_0 \cup \partial Q_{(0,0)} + m_1\omega_1 + m_2\omega_2)$ and $D_k + m_1\omega_1 + m_2\omega_2$ (m_1, m_2 are integers) are occupied by materials of conductivities λ_0 and λ, respectively. The conductivity of the inclusions λ is expressed relative to λ_0. Hence, the conductivity of the matrix can be taken as unity ($\lambda_0 = 1$). The local potential potential $u(z)$ in $Q_{(0,0)}$ satisfies the conjugation conditions:

$$u^+(t) = u^-(t), \quad \frac{\partial u^+}{\partial n}(t) = \lambda \frac{\partial u^-}{\partial n}(t) \text{ on } \partial D_k = \{t \in \mathbb{C} : |t - a_k| = r\}, \quad k = 1, 2, \ldots, N, \tag{19}$$

where $(\partial/\partial n)$ is the outward normal derivative and, for instance,

$$u^+(t) := \lim_{\substack{z \to t, \\ z \in D_0}} u(z), \qquad u^-(t) := \lim_{\substack{z \to t, \\ z \in D_k}} u(z). \tag{20}$$

The potential $u(z)$ satisfies the quasi-periodicity conditions

$$u(z + \omega_1) = u(z) + \Omega_1, \quad u(z + \omega_2) = u(z) + \Omega_2. \tag{21}$$

Here, the function $u(z)$ is harmonic in $Q_{(0,0)}$ except ∂D_k $(k = 1, 2, \ldots, N)$, the circles ∂D_k are orientated in the clockwise direction. Equations (19) model the perfect contact between matrix and inclusions. Equations (21) mean that the external field has the gradient (Ω_1, Ω_2) in the coordinates based on the vectors ω_1 and ω_2. In order to

determine the effective conductivity tensor Λ_e it is sufficient to solve problem (19), (21) with two linear independent vectors (Ω_1, Ω_2).

The effective conductivity tensor Λ_e of the composite considered has the following structure:

$$\Lambda_e = (1 + 2\rho v)\mathbf{I} + 2\rho v \sum_{k=1}^{\infty} \mathbf{P}_k v^k, \tag{22}$$

where $v = (N\pi r^2/|Q_{(0,0)}|)$ is the concentration of the disks in the cell $Q_{(0,0)}$, $\left|Q_{(0,0)}\right|$ is the area of $Q_{(0,0)}$, $\mathbf{I}$ is the identity tensor,

$$\mathbf{P}_k = \begin{pmatrix} \mathrm{Re}A_k & \mathrm{Im}\,A_k \\ \mathrm{Im}\,A_k & C_k \end{pmatrix},$$
$$A_k = \left|Q_{(0,0)}\right|^k \sum_{n_1 \ldots n_p} B^{(k)}_{n_1 \ldots n_p} e_{n_1 \ldots n_p}(\omega_1, \omega_2), \tag{23}$$

The constants $B^{(k)}_{n_1 \ldots n_p}$ depend only on k, ρ and $n_1, \ldots, n_p$. Here, $n_j = 2, 3, \ldots$; $k = 1, 2, \ldots$. The values C_k have an analogous form. The terms $e_{n_1 \ldots n_p}(\omega_1, \omega_2)$ only depend on the centers of inclusions a_k in the representation (22)–(23) of Λ_e. Few first coefficients A_k have the form

$$A_1 = \frac{\rho}{\pi} e_2, \quad A_2 = \frac{\rho^2}{\pi^2} e_{22}, \quad A_3 = \frac{1}{\pi^3}\left[-2\rho^2 e_{33} + \rho^3 e_{222}\right],$$
$$A_4 = \frac{1}{\pi^4}[3\rho^2 e_{44} - 2\rho^3(e_{332} + e_{233}) + \rho^4 e_{2222}],$$
$$A_5 = \frac{1}{\pi^5}[-4\rho^2 e_{55} + \rho^3(3e_{442} + 6e_{343} + 3e_{244}) + 2\rho^4(e_{3322} + e_{2332} + e_{2233}) + \rho^5 e_{22222}],$$
$$A_6 = \frac{1}{\pi^6}[5\rho^2 e_{66} - \rho^3(4e_{255} + 12e_{354} + 12e_{453} + 4e_{552}) + \rho^4(3e_{2244} + 6e_{2343} + 4e_{3333} + 3e_{2442} + 6e_{3432} + 3e_{4422}) - 2\rho^5(e_{22233} + e_{22332} + e_{23322} + e_{33222}) + \rho^6 e_{222222}], \tag{24}$$

where the argument (ω_1, ω_2) is omitted. It follows from (23) that each coefficient A_k involves a set of $e_{n_1 \ldots n_p}(\omega_1, \omega_2)$. The mapping $k \mapsto (n_1, \ldots, n_p)$ has been described precisely in [6,7]. In particular, the following conditions are fulfilled:

(i) $n_1 + \cdots + n_p = 2k$;
(ii) $n_j \le k - p$ $(j = 1, 2, \ldots, p)$;
(iii) $n_j \ge 2$ $(j = 1, 2, \ldots, p)$.

Hence, in order to calculate the tensor Λ_e up to $O(v^{L+1})$, we have to find A_k $(k = 1, 2, \ldots, L-1)$. Therefore, we have to calculate a finite number of the generalized Eisenstein–Rayleigh sums $e_{n_1 \ldots n_p}(\omega_1, \omega_2)$.

4. Representative cell

4.1. *General equations*

Consider a large fundamental region $Q'_{(0,0)}$ constructed by the fundamental translation vectors ω'_1 and ω'_2. Let $Q'_{(0,0)}$ contain N' non-overlapping circular disks D'_k of the radius r with the centers $a'_k \in Q'_{(0,0)}$ $(k = 1, 2, \ldots, N')$. Let Λ'_e be the effective conductivity tensor of the composite material represented by the region $Q'_{(0,0)}$ with inclusions D'_k. We are interested in the following question: To replace $Q'_{(0,0)}$ by another small cell $Q_{(0,0)}$ which contains inclusions $D_k = \{z \in \mathbb{C} : |z - a_k| < r\}$ $(k = 1, 2, \ldots, N)$ and which has an effective conductivity tensor Λ_e closed to Λ'_e. We assume that the concentration v of the inclusions in both materials is the same. Closeness is defined by the accuracy $O(v^{L+1})$ for the difference $\Delta\Lambda_e = \Lambda_e - \Lambda'_e$ with prescribed L. We say that $Q_{(0,0)}$ is a representative cell for the region $Q'_{(0,0)}$ with the accuracy $O(v^{L+1})$ if $\Delta\Lambda_e = O(v^{L+1})$. We say that $Q_{(0,0)}$ is the minimal representative cell for the region $Q'_{(0,0)}$ if $Q_{(0,0)}$ is a representative cell with minimal possible area $|Q_{(0,0)}|$. For brevity, we further call the minimal representative cell by the representative cell. The existence of the representative cell is evident, since in the worst case one can take $Q_{(0,0)} = Q'_{(0,0)}$.

We adopt the designations (22), (23) for the representative cell. Consider the corresponding formulas for Λ'_e

$$\Lambda'_e = (1 + 2\rho v)\mathbf{I} + 2\rho v \sum_{k=1}^{\infty} \mathbf{P}'_k v^k, \tag{25}$$

$$A'_k = \left|Q'_{(0,0)}\right|^k \sum_{n_1 \ldots n_p} B^{(k)}_{n_1 \ldots n_p} e_{n_1 \ldots n_p}(\omega'_1, \omega'_2). \tag{26}$$

Note that the coefficients $B^{(k)}_{n_1 \ldots n_p}$ has the same form in (23) and (26). $\Delta\Lambda_e$ is of order $O(v^{L+1})$ if $A'_k = A_k$ for $k = 1, 2, \ldots, L-1$. Therefore, $\Delta\Lambda_e$ is of order $O(v^{L+1})$ if and only if

$$\left|Q_{(0,0)}\right|^k e_{n_1 \ldots n_p}(\omega_1, \omega_2) = \left|Q'_{(0,0)}\right|^k e_{n_1 \ldots n_p}(\omega'_1, \omega'_2) \tag{27}$$

for $k = 1, 2, \ldots, L-1$ and corresponding sets of the numbers $n_1, \ldots, n_p$. According to our definition $Q_{(0,0)}$ is a representative cell for the region $Q'_{(0,0)}$ with the accuracy $O(v^{L+1})$ if and only if the relations (27) are fulfilled.

One can consider (27) as a system of equations with respect to $\omega_1, \omega_2, a_1, a_2, \ldots, a_N$ including the unknown number N with the restriction $|a_j - a_m| \le 2r$ $(j \ne m)$. One can assume that one of the centers, say a_N, lies at the origin, since geometrically any cell is determined up to translation. The fundamental region $Q_{(0,0)}$ as well as the translation vectors ω_1, ω_2 can be chosen by infinitely many ways [10]. For any doubly periodic structure on the plane it is always possible to construct a pair ω_1, ω_2 such that $\omega_1 > 0$ and $\operatorname{Im}\tau > 0$.

The area of $Q_{(0,0)}$ is calculated by ω_1 and ω_2

$$\left|Q_{(0,0)}\right| = \omega_1^2 \operatorname{Im}\tau. \tag{28}$$

On the other hand, we also have

$$\left|Q_{(0,0)}\right| = \frac{N\pi r^2}{v} \tag{29}$$

that yields the formula

$$\omega_1 = \sqrt{\frac{N\pi r^2}{v\,\mathrm{Im}\,\tau}}. \tag{30}$$

In order to construct the representative cell with the prescribed accuracy $O(v^{L+1})$, we propose to solve the system (27) with fixed L increasing the number of inclusions in the cell N from 1 to N'. Then N is fixed in each step of the study of (27).

Applying (18) and (28) we rewrite (27) in the form

$$(\mathrm{Im}\,\tau)^k e_{n_1\ldots n_p}(1,\tau) = \left|Q'_{(0,0)}\right|^k e_{n_1\ldots n_p}(\omega'_1,\omega'_2), \quad k = 1,2,\ldots,L-1. \tag{31}$$

We can consider (31) as a system with respect to $\tau, a_1, a_2, \ldots, a_{N-1}$ ($a_N = 0$) with the restriction $\left|a_j - a_m\right| \geq 2r$ ($j \neq m$). The right-hand part of (31) is known. If we know a solution of (31), we can calculate ω_1 from (30).

It is also possible to state the problem of the representative cell with prescribed form of the cell $Q_{(0,0)}$. Let us consider the case when $Q_{(0,0)}$ is a rectangle. Then $\tau = i\alpha$, where α is positive and (30) implies

$$\omega_1 = \sqrt{\frac{N\pi r^2}{\alpha v}}. \tag{32}$$

Equation (31) becomes

$$\alpha^k e_{n_1\ldots n_p}(1,i\alpha) = \left|Q'_{(0,0)}\right|^k e_{n_1\ldots n_p}(\omega'_1,\omega'_2), \quad k = 1,2,\ldots,L-1. \tag{33}$$

It is hard to investigate analytically the systems (31), (33) in general form, since it is difficult to extract independent equations from the sets (31), (33). In the next subsection simple examples of (33) are considered.

4.2. *Two inclusions in the representative cell*

Consider two inclusions in a rectangular cell ($N=2$). In this case, the positions of the inclusions are determined by one complex parameter $a = a_2 - a_1$. By direct calculation one can check that

$$e_n(1,\tau) = \begin{cases} \dfrac{1}{2}(S_n(1,\tau) + E_n(a;1,\tau)) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{34}$$

$$e_{mn}(1,\tau) = e_m(1,\tau)\overline{e_n(1,\tau)}, \quad e_{mnp}(1,\tau) = e_m(1,\tau)\overline{e_n(1,\tau)}e_p(1,\tau), \tag{35}$$

and so on. Therefore, instead of the general Eisenstein–Rayleigh sums in (33) it is sufficient to consider equations with simple sums

$$\alpha^k e_k(1, i\alpha) = \left|Q'_{(0,0)}\right|^k e_k(\omega'_1, \omega'_2), \quad k = 2, 4, \dots. \tag{36}$$

Substitution of (34) into (36) in the case $\tau = i\alpha$ yields

$$\alpha^k[S_k(1, i\alpha) + E_k(a; 1, i\alpha)] = p_k, \quad k = 2, 4, \dots, \tag{37}$$

where

$$p_k = 2\left|Q'_{(0,0)}\right|^k e_k(\omega'_1, \omega'_2)$$

are known constants. Equation (37) with $k = 2, 4$ becomes

$$\alpha^2[S_2(1, i\alpha) + E_2(a; 1, i\alpha)] = p_2, \quad \alpha^4[S_4(1, i\alpha) + E_4(a; 1, i\alpha)] = p_4. \tag{38}$$

Using the relations [8,10]

$$E_4(z; 1, i\alpha) = \frac{1}{6}\frac{\mathrm{d}^2}{\mathrm{d}z^2} E_2(z; 1, i\alpha),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} E_2(z; 1, i\alpha) = 6(E_2(a; 1, i\alpha) - S_2(1, i\alpha))^2 - \frac{1}{2} g_2(1, i\alpha)$$

and (8) we obtain

$$E_4(z; 1, i\alpha) = (E_2(a; 1, i\alpha) - S_2(1, i\alpha))^2 - 5S_4(1, i\alpha). \tag{39}$$

Then the second equation (38) is transformed to the following:

$$\alpha^4[(E_2(a; 1, i\alpha) - S_2(1, i\alpha))^2 - 4S_4(1, i\alpha)] = p_4. \tag{40}$$

We express $E_2(a; 1, i\alpha)$ from the first equation (38)

$$E_2(a; 1, i\alpha) = \frac{p_2}{\alpha^2} - S_2(1, i\alpha) \tag{41}$$

and substitute it in (40)

$$\alpha^4\left[\left(\frac{p_2}{\alpha^2} - 2S_2(1, i\alpha)\right)^2 - 4(S_4(1, i\alpha)\right] = p_4. \tag{42}$$

The latter equation is real, since S_2 and S_4 are real for rectangular arrays. Hence, we have obtained the real number equation (42) with respect to real unknown α.

Let us consider a numerical example with $p_2 = 10$, $p_4 = 50$, $r = 0.15$, $v = 0.3$. Equation (42) has the solution $\alpha = 0.820$. Substituting α in (41) and solving the obtained equation with respect to a we get $a = 0.331$. Then (32) implies $\omega_1 = 0.758$. Therefore, the representative cell is described by the fundamental vectors $\omega_1 = 0.934$, $\omega_2 = i0.820$ with two inclusions with the centers $a_1 = 0$ and $a_2 = 0.331$.

4.3. *Three inclusions in the representative cell*

Consider now three inclusions in the cell ($N = 3$). In this case the positions of the inclusions are determined by two complex parameter a_1 and a_2 ($a_3 = 0$). It follows from equation (30) that

$$\omega_1^2 = c_0(\operatorname{Im}\tau)^{-1}, \tag{43}$$

where $c_0 = 3\pi r^2/v$. Equation (31) becomes

$$(\operatorname{Im}\tau)^k e_{2k}(1,\tau) = p_{2k}, \quad k = 1,2,3, \tag{44}$$

where e_{2k} are calculated by (15). For numerical computations it is convenient to use formulas (12) involving the Weierstrass function $\mathcal{P}(z)$ and its derivatives. Then $e_{2k}(1,\tau)$ become

$$e_2(1,\tau) = S_2(1,\tau) + \frac{2}{9}(\mathcal{P}(a_1) + \mathcal{P}(a_2) + \mathcal{P}(a_1 - a_2)), \tag{45}$$

$$e_4(1,\tau) = -3S_4(1,\tau) + \frac{2}{9}\big(\mathcal{P}^2(a_1) + \mathcal{P}^2(a_2) + \mathcal{P}^2(a_1 - a_2)\big), \tag{46}$$

$$e_6(1,\tau) = \frac{1121}{3}S_6(1,\tau) + 6S_4(1,\tau)(\mathcal{P}(a_1) + \mathcal{P}(a_2) + \mathcal{P}(a_1 - a_2)) - \frac{22}{45}\big(\mathcal{P}^3(a_1) + \mathcal{P}^3(a_2) + \mathcal{P}^3(a_1 - a_2)\big). \tag{47}$$

Here we use the relations (12) and the following formulas from [8,10]

$$E_4(z) = \frac{1}{6}\mathcal{P}''(z) = \mathcal{P}^2(z) - 5S_4(1,\tau), \tag{48}$$

$$E_6(z) = -\frac{11}{5}\mathcal{P}^3(z) + 27S_4(1,\tau)\mathcal{P}(z) + 560S_6(1,\tau). \tag{49}$$

Therefore, we arrive at three equations (44) where e_{2k} have the form (45)–(47) with respect to three unknowns a_1, a_2 and τ.

Consider a numerical example in which the large cell $Q'_{(0,0)}$ (see figure 1) with $N' = 60$ inclusions of the radius $r = 0.12$ and the concentration $v = 0.1$ is determined by the translation vector $\omega'_1 = 4$, $\omega'_2 = 4i$. In this case $p_2 = 0.78 - 0.66i$, $p_4 = -2.15 + 2.27i$, $p_6 = -6.28 - 51i$. The cell $Q'_{(0,0)}$ is replaced by a smaller cell $Q_{(0,0)}$ with $N = 3$ inclusions. In order to find parameters of $Q_{(0,0)}$ we solve the system (44). One of the

solutions has the form $a_1 = -0.92$, $a_2 = -0.36 + 0.36i$, $\tau = 0.06 + 0.39i$. Then (30) yields $\omega_1 = 1.08$. The cell $Q_{(0,0)}$ is presented in figure 2.

5. Conclusion and discussion

In the present article a rigorous theory of the representative cell in mechanics of periodic composites is proposed. We restrict ourselves by conductivity of two-dimensional two-component composite materials made from a collection of non-overlapping, identical, circular disks, arbitrarily embedded in a matrix. The definition of the representative cell is based on the representation of the effective conductivity tensor (22)–(23). We say that the cells $Q_{(0,0)}$ and $Q'_{(0,0)}$ are equivalent if they have the same basic elements in the representation (22)–(23). Thus all the composites are divided onto classes of equivalences. The minimal size cell in each class is called the representative cell. The basic elements of (22)–(23) are expressed in terms of the generalized Eisenstein–Rayleigh sums (14).

The problem of determination of the representative cell is reduced to the finite system of equation (31) with respect to parameters of the representative cell. Partial cases of this system are considered.

We investigate here the problem of numerical solution to the system (31) with $N=2$ and $N=3$. We can give some remarks about the general system (31). First, it is evident that it has infinite number of solutions, since any doubly periodic structure is determined by infinite number of the pairs of the fundamental translation vectors. Moreover, as it follows from relations (34)–(35), the system (31) can contain redundant equations. Note also that, for rectangular cell, a symmetric map of the location of inclusions could not change the effective conductivity tensor. Hence, if the fundamental translation vectors are fixed, the system (31) with respect to $a_1, a_2, \ldots, a_{N-1}$ can have non-unique solution.

In the present study we discussed the conductivity two-dimensional problem with circular inclusions. Now we briefly explain why the method can be developed and applied to other problems of the theory of composites.

First, let us consider the same problem but with inclusions having another shape S. Any plane domain S can be approximated by packing disks of the radius r. This approximation can be expressed by appropriate conditions on the centers of the packing disks $b_1, b_2, \ldots, b_P$. We write them in the form of the constraints on b_j

$$b_j - b_1 = B_j e^{i\psi}, \quad j = 3, 4, \ldots, P, \quad |B_2| = |b_2 - b_1|. \tag{50}$$

Here, the constants $B_2, B_3, \ldots, B_P$ are given, $\psi = \arg(B_2/(b_2 - b_1))$. The constrains (50) mean that the points $b_1, b_2, \ldots, b_P$ are tied and may only translate and rotate as a stiff body. We can replace all inclusions (say M inclusions per cell) by a set of points $a_1, a_2, \ldots, a_N$ divided onto M subsets each of them contains P points. We assume that points of each subset satisfy the constrains (50), i.e., each subset of the disks approximates an inclusion of the form S. These constrains on $a_1, a_2, \ldots, a_N$ should be added to equations (31) in order to obtain a system of equations corresponding to the representative cell with M inclusions of the shape S.

The proposed method could also be applied to elastic problems. In addition to the classical Eisenstein functions, we should consider the following series [13,14]:

$$E_{pq}(z) = \sum_{m_1, m_2} = \sum_{m_1, m_2} (z - m_1 - im_2)^{-p} \overline{(z - m_1 - im_2)^{-q}}. \tag{51}$$

The lattice sums (51) were presented in [14] in rapidly convergent forms by Fourier transform methods.

Berdichevskij [15] constructed three-dimensional counterparts of the elliptic functions which could be used for three-dimensional conductivity and elasticity problems. Huang [16] proposed exact integral formulas for three-dimensional lattice sums. The examples therein show that simple quadrature rules with modest numbers of nodes yield highly accurate results. One can find a review of the various numerical methods to calculate three-dimensional lattice sums in [16].

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