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The $\mathcal{P}$–function of Weierstrass is not topologically conjugated to exponent

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Abstract

We discuss Arnold's problem on the topologically elementary functions. We prove that the $\mathcal{P}$–function of Weierstrass cannot be homeomorphically conjugated to the exponential function.

1 Introduction

One can hear first from the secondary school that it is impossible to express roots of the algebraic equations of degree 5 or higher in terms of the coefficients using only arithmetic operations and radicals. This assertion were proved by Ruffini in 1799 with minor gaps (see historical note [3]) and it is known in our days as the Abel (–Ruffini) theorem.

In 1963 Vladimir Igorevich Arnold gave the special course *Abel's theorem* for pupils of the College of the Moscow State University. Latter V. B. Alekseev prepared the book [2] according to this course. In 1963-64 V. I. Arnold has proved that equation $x^5 + ax + 1 = 0$ cannot be solved in wider sense, namely the roots of this equation cannot be presented as a topologically elementary function $x(a)$. In what follows elementary functions are those which can be obtained from the basic elementary functions (polynomials, exp, log, trigonometric functions and root extractions) by finite number of arithmetic operations and compositions. The notation of the topologically elementary function will be given below. In 1963 V. I. Arnold had stated the following question. Are the elliptic integral and the Weierstrass $\mathcal{P}$–function topologically elementary? He also proposed a plan of the long proof of this conjecture not realized yet. One can find discussion devoted to this question and to many other interesting facts in [2]–[8].

We recall that the elliptic integral is defined as follows [1]

$$u(w) = \int_w^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \tag{1}$$

where g_2 and g_3 are given constants. The Weierstrass $\mathcal{P}$–function with the periods ω_1, ω_2 ($Im\ \omega_2/\omega_1 > 0$) can be defined as the series

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m^2+n^2 \neq 0} \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right). \tag{2}$$

It satisfies the differential equation

$$dz = -\frac{d\mathcal{P}(z)}{\sqrt{4[\mathcal{P}(z)]^3 - g_2 \mathcal{P}(z) - g_3}}, \tag{3}$$

where g_2 and g_3 are related with the periods ω_1 and ω_2. Comparing (1) and (3) one can see that the elliptic integral $u(w)$ is the inverse function to $\mathcal{P}(z)$.

In the present note we demonstrate that the $\mathcal{P}$–function is not topologically conjugated to exponent. This proof constitutes the main part of the theorem from [9], where we prove that the $\mathcal{P}$–function is not topologically elementary.

At the beginning we give the fundamental definitions and notations used in the present note. It is convenient to use four copies $\mathbb{C}_j$ ($j = 1, 2, 3, 4$) of the complex plane $\mathbb{C}$.

Definition 1 *A function $\widetilde{f}\colon \mathbb{C}_3 \to \mathbb{C}_4$ is called the topologically conjugated to $f(z)$ if there exist homeomorphisms h, k of the complex plane such that the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{C}_1 & \xrightarrow{f} & \mathbb{C}_2 \\ h \downarrow & & \downarrow k \\ \mathbb{C}_3 & \xrightarrow{\widetilde{f}} & \mathbb{C}_4 \end{array} \tag{4}$$

Definition 2 *A function* $\widetilde{f}\colon \mathbb{C}_3 \to \mathbb{C}_4$ *is called the topologically elementary if there exist an elementary function* f *topologically conjugated to* $\widetilde{f}$*.*

In order to distinguish the behavior of the curves at infinity we introduce the following definition.

Definition 3 *Let a continuous curve* γ *is defined by the parametrization* $x \mapsto g(x)$*, where* $0 \le x < 1$*,* $g(x) \in \mathbb{C}$*. One says that the curve* γ *tends to infinity if the following limit exists*

$$\lim_{x\to 1-0} g(x) = \infty. \tag{5}$$

In the same time, one says that γ *goes by infinity if for any* $R > 0$ *there exist points of* γ *which do not lie in the disk* $|z| < R$*.*

Theorem 4 *The exponential function and the Weierstrass* $\mathcal{P}$*–function are not topologically conjugated.*

Proof is given by reductio ad absurdum. Assume that the Weierstrass $\mathcal{P}$–function is topologically conjugated to the exponent $\exp(z) = e^z$, i.e., the diagram (4) is commutative with $f = \exp$, $\widetilde{f} = \mathcal{P}$ and some k i h. In other words $k \circ \exp = \mathcal{P} \circ h$. The function exp transforms the half–plane $D_1 = \{z \in \mathbb{C}_1 : Re\ z > 0\}$ onto $D_2 = \{z \in \mathbb{C}_2 : |z| > 1\}$. Introduce $D_3 = h(D_1)$, $D_4 = k(D_2)$ and denote by ∂D_3, ∂D_4 the boundaries of these domains. The curve ∂D_3 goes by infinity in the both directions and divides the plane $\mathbb{C}_3$ onto two domains, since h is homeomorphism D_1 onto D_3. The curve ∂D_4 divides $\mathbb{C}_4$ onto two domains. Moreover, the domain D_4 contains infinity and its complement to $\mathbb{C}_4$ is a bounded domain. Therefore, $\gamma_4 = (a, +\infty)$ for sufficiently large positive a entirely lies in the domain D_4.

According to the definition γ_4 tends to infinity. Introduce the curve $\gamma_2 = k^{-1} \circ \gamma_4$ on the plane $\mathbb{C}_2$, the curve $\gamma_1 = \exp^{-1} \circ \gamma_2 = \log \circ \gamma_2$ in the strip $\{z \in \mathbb{C}_1 : 0 \le Im\ z < 2\pi\}$, and the curve $\gamma_3 = h \circ \gamma_1$ on the plane $\mathbb{C}_3$. The curves γ_1, γ_2 and γ_3 go by infinity, since h and k are homeomorphisms of the complex plane and they are not necessary continuous at infinity. The diagram (4) yields $\gamma_4 = k \circ \exp \circ \gamma_1 = \mathcal{P} \circ h \circ \gamma_1$.

We construct a doubly periodic lattice with the periods $\frac{1}{2}\omega_1$, ω_2 on the plane $\mathbb{C}_3$. Consider the parallelograms $\Pi_{(m,n)} = \Pi_{(0,0)} + \frac{m}{2}\omega_1 + n\omega_2$ with $(m, n) \in \mathbb{Z}^2$, where $\Pi_{(0,0)}$ has the vertices $-\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2$, $-\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$, $\frac{1}{2}\omega_2$, $-\frac{1}{2}\omega_2$.

The $\mathcal{P}$–function is univalent in each $\Pi_{(m,n)}$ and has the double poles at one of the sides of each parallelogram (see formula (2) and Figure 1). We note that the parallelograms $\Pi_{(m,n)}$ are one half of the periodicity parallelograms, say $Q_{(2m,n)}$, of the $\mathcal{P}$–function. The Weierstrass function maps any parallelogram $\Pi_{(m,n)}$ onto the plane with a cut (see Figure 2).

The curve γ_3 goes by infinity intersecting the sides of $Q_{(2m,n)}$ at points denoted by W. The range of these points $\mathcal{P}(W)$ must belong to the supporter of the curve $\gamma_4 = \mathcal{P} \circ \gamma_3$ and simultaneously to $L_1 \cup L_3 \cup L_4$. This yields the contradiction.

The theorem is proved.

The topological non–elementary of the roots of $x^5 + ax + 1 = 0$ discussed above was presented in 1960 years by V. I. Arnold to pupils of the secondary school of the USSR (exceptional school created by A. N. Kolmogorov for gifted children). The presented here theorem could be also treated as understandable by a gifted schoolboy may be up to the Weierstrass function. However, I have to say with regret that for a schoolboy from 1960 years. Unfortunately, the complex numbers are out of the secondary school program. The complex analysis with traditional applications to mechanics of continuum and to fluids is not popular in high education. Absolvent of the mathematics does not hear about the Weierstrass function. Usually the complex analysis is given in Polish Universities only in the second semester of IV years that complicates to prepare the magister diploma in this topics. As a result one can meet scientific papers with complicated solutions of the problems which can be easily solved by conformal mappings or papers with chaos in the definitions of the logarithmic brunches.

I am grateful to V. I. Arnold for explanations concerning the problem of topological non–elementary functions and for the literature.

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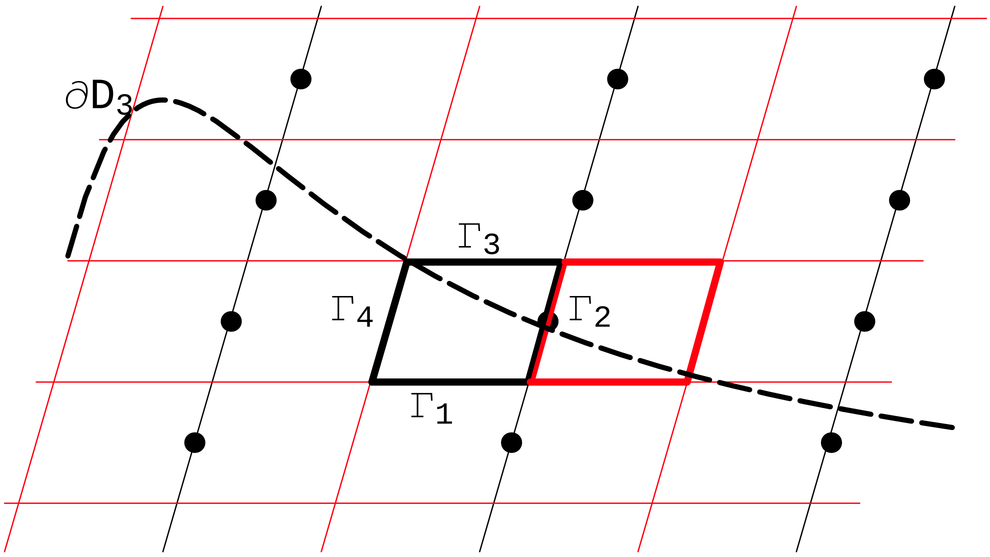

Figure 1: Complex plane $\mathbb{C}_3$. Univalent parallelogram of the $\mathcal{P}$–function with sides Γ_1, Γ_2, Γ_3, Γ_4. The side Γ_2 contains the pole of the $\mathcal{P}$–function. Two univalent parallelograms generate a periodicity parallelogram of the $\mathcal{P}$–function

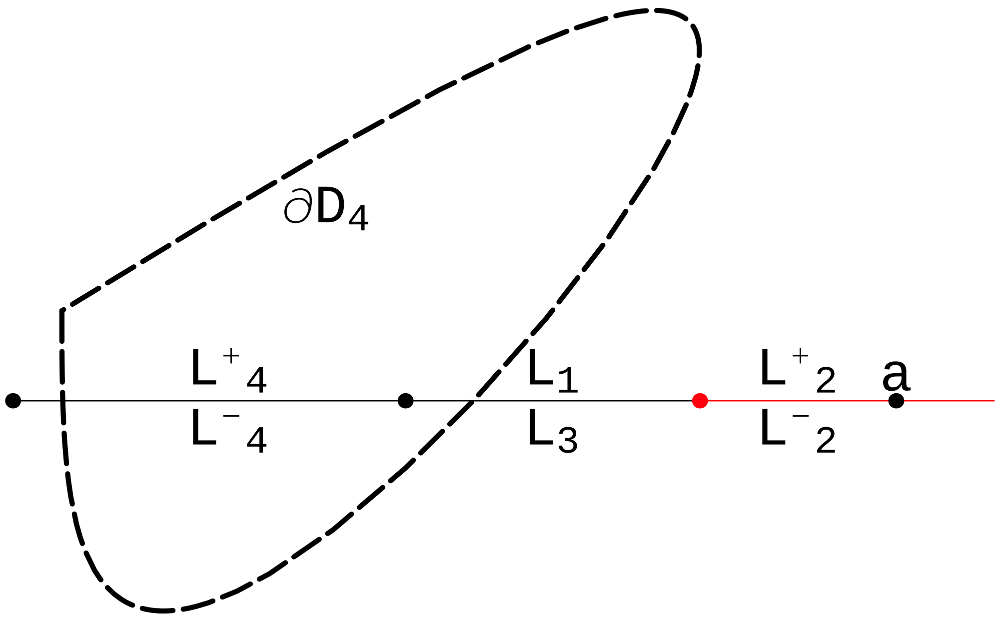

Figure 2: Plane $\mathbb{C}_4$. The segments L_k with up and down boundaries are ranges of the sides Γ_k, i.e., $L_k = \mathcal{P}(\Gamma_k)$ for $k = 1, 2, 3, 4$